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ENGINEERING METHODS FOR ANALYSING STRENGTH AND RIGIDITY

Г. С. ГЛУШКОВ

ИНЖЕНЕРНЫЕ МЕТОДЫ РАСЧЕТОВ НА ПРОЧНОСТЬ И ЖЕСТКОСТЬ

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The Greek Alphabet

Α α	Alpha	Ι ι	Iota	Ρ ρ	Rho
Β β	Beta	Κ ϰ	Kappa	Σ σ	Sigma
Γ γ	Gamma	Λ λ	Lambda	Τ τ	Tau
Δ δ	Delta	Μ μ	Mu	Υ υ	Upsilon
Ε ε	Epsilon	Ν ν	Nu	Φ φ	Phi
Ζ ζ	Zeta	Ξ ξ	Xi	Χ χ	Chi
Η η	Eta	Ο ο	Omicron	Ψ ψ	Psi
Θ θ	Theta	Π π	Pi	Ω ω	Omega

На английском языке

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Preface

Scientific progress is generally accompanied by an increasing accuracy of theoretical methods and by a greater generalization of the symbolics and mathematical apparatus.

As we trace the development of the theory of higher moments, we are convinced that the order of moments involved in practical and theoretical application has been growing gradually, and with it the variety of moments.

Indeed, long before our era Euclidian geometry already dealt with areas of plane figures; actually areas are moments of zero order of geometrical figures. Leonardo da Vinci initiated the use of moments of the first order by introducing static moments of plane figures and point loads.

The next step was taken only in the XIX century, when the theory of moments of inertia of plane figures (moments of the second order) was developed by Culman, Lamé and other scientists. Moments of the zero, first and second order were the only types of moments employed in engineering and in structural mechanics for a long time.

Although the concepts of higher moments and factorial moments were discussed in mathematical literature and in articles on strength of materials, no mention was made of their application in engineering practice until the author of this book pointed out such a possibility in his paper published in 1920. In this work the uniaxial moments of the second and third orders were used. Later such concepts as co-moments (which are a variant of biaxial moments), uniaxial and biaxial moments of vector quantities, etc. were introduced by other authors.

As a result, the theory of moments developed into a specific branch of mathematics which became a useful tool for solving complicated problems in structural mechanics. A number of Soviet scientists developed the so-called moment-operational method, which has proved to be extremely efficient for solving various problems of modern engineering. These problems arise when non-linear elasticity must be taken into account, when precise designing of non-uniform structural elements is required, when the loading of a structure is essentially non-uniform, etc.

The moment-operational method has been widely employed for compiling manuals containing numerous tables and formulas for designing beams, arches and frames. A number of scientific research works and textbooks present the theory of moments of higher order and give examples of problems solved by the moment-operational method.

The knowledge of the theory of higher moments as well as of the moment-operational method will serve to extend the field of their application to new problems.

PART I

Moments of Higher Order: Theory and Application

Chapter I

Theory of Moments

1. GENERAL CONCEPT OF MOMENTS OF AREA

The concept of moments of area applies to the cross-sectional areas of beams, to loading areas of beams subjected to continuous loads (forces and moments), to diagrams allowing for variability of temperature, density, modulus of elasticity, etc., to all kinds of plane figures referred to in the theory of resistance of materials.

Let x be directed along the axis of a straight beam or along the tangent to the axis of a curved beam, and the principal axes y and z be located in the cross-sectional plane F (Fig. 1.1); then the integral

$$I_{z^m} = \int\limits_F y^m \, dF \qquad (1.1)$$

will be the m-th moment of area F with respect to axis z, and the integral

$$I_{y^n} = \int\limits_F z^n \, dF \qquad (1.2)$$

will be the n-th moment of the same area with respect to axis y.

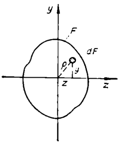

Fig. 1.1

Assuming $m, n = 1, 2, 3, \ldots$, from formulas (1.1) and (1.2) we obtain moments of the first, second, third, etc. orders of area F with respect to axes z and y lying in plane F.

For example, the first order moment

$$I_{y^1} = S = \int\limits_F z \, dF \qquad (1.3)$$

is known as the static moment of area F with respect to axis y.

The second order moment

$$I_{y^2} = I_y = \int_F z^2\, dF \tag{1.4}$$

is the moment of inertia of area F with respect to axis y.

The third order moment is

$$I_{y^3} = I_3 = \int_F z^3\, dF \tag{1.5}$$

etc.

All the enumerated moments are uniaxial moments of area F with respect to axis y. The uniaxial moments of any order with respect to axis z are calculated in a similar way by formula (1.1).

Moments may be determined not only with respect to one axis, but also with respect to two, three, or more axes, resulting in biaxial, triaxial, . . ., multiaxial moments.

A general expression for biaxial moments of the (m, n)-th order of area F with respect to axes z and y is given by the following integral*

$$I_{z^m y^n} = \int_F y^m z^n\, dF \tag{1.6}$$

which is a moment of the $(m + n)$-th order, i.e., a moment of the m-th order with respect to axis z and of the n-th order with respect to axis y.

For example, the following expression

$$I_{z^3 y^4} = \int_F y^3 z^4\, dF \tag{1.7}$$

is a biaxial moment of area F with respect to axes z and y the moment of the third order being taken about axis z and the moment of the fourth order being taken about axis y.

More complicated designations, such as I_{z^m} and I_{y^m}, should be used when moments with respect to different axes are considered, because both the axis and the order must be indicated. Simpler designations, such as I_1, I_2, I_3, . . ., I_m which indicate the order only, may be used when the moments involved are uniaxial.

Biaxial moments may be calculated with respect to mutually perpendicular axes as well as with respect to axes intersecting at an arbitrary angle α (Fig. 1.2)

---

* The general expression for a biaxial moment can be given in the form of the following integral $I = \int_F f(z, y)\, dF$, where $f(z, y)$ is an algebraic polynomial of coordinates z and y.

$$I_{u^m v^n} = \int_F v^m u^n \, dF \tag{1.8}$$

In particular, the axes may be mutually parallel (Fig. 1.3)

$$I_{y_1^m y_2^n} = \int_F z_1^m z_2^n \, dF \tag{1.9}$$

Formulas (1.8) and (1.9) are expressions for moments with respect to two different axes, the moment being of the order m with respect to one of the axes and of the order n with respect to the other. If one of the moments is assumed to have a zero order, say, $n = 0$, then an expression is obtained for a uniaxial moment as a particular case of biaxial moments.

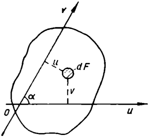

Fig. 1.2

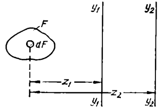

Fig. 1.3

Multiaxial moments of higher order of area F with respect to three or more axes are determined in a similar way.

For determining a multiaxial moment the element dF of the area is multiplied by the distance from the corresponding axis raised to powers equal to the order of the moment. As an example, in the case of three mutually parallel axes y_1-y_1, y_2-y_2, and y_3-y_3 (Fig. 1.4), the moment

$$I_{y_1^m y_2^n y_3^k} = \int_F z_1^m z_2^n z_3^k \, dF \tag{1.10}$$

is a moment of the $(m + n + k)$-th order, i.e., of order m with respect to axis y_1-y_1, of order n with respect to axis y_2-y_2, and of order k with respect to axis y_3-y_3.

Similar to moments of higher orders determined by formula (1.1), a moment of any order of area F can be determined with respect to the point O (a polar moment). To determine the moment of the n-th order of area F (Fig. 1.5) with respect to point O (any point in the plane may be chosen for the pole) the distance from the pole in the integrand should be raised to power n corresponding to the order of the moment

$$I_{O^n} = \int_F \rho^n \, dF \tag{1.11}$$

Taking $n = 2$ in formula (1.2), $m = n = 1$ in (1.6) and $n = 2$ in (1.11) we obtain expressions for axial, centrifugal and polar moments of inertia of area F.

Multiaxial moments of higher order may be determined with respect to several axes, some of these axes lying in the plane of the

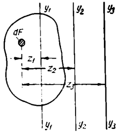

Fig. 1.4

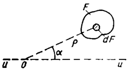

Fig. 1.5

area and the others being perpendicular to that plane. In this case the so-called polar-axial moments are obtained; for example, the following expression

$$I_{z^m O^n} = \int_F y^m \rho^n \, dF \tag{1.12}$$

is a polar-axial moment of the $(m + n)$-th order of area F (Fig. 1.1), determined as a moment of the n-th order with respect to the pole O and of the m-th order with respect to axis z-z.

The following polar-axial moments of the third order can serve as an example*

$$U_z = \int_F y\rho^2 \, dF; \quad U_y = \int_F z\rho^2 \, dF \tag{1.13}$$

---

* The quantities U_z and U_y in formulas (1.13) are expressed in terms of uniaxial moments of the third order if $\rho^2 = z^2 + y^2$, where z and y are the coordinates of dF.

The above mentioned examples do not cover all possible types of moments. However, other types of moments will not be considered, because they are not widely used in engineering.

Moments of higher order are used in determining moments of uniformly distributed loads, of uniformly varying loads, and of loads having areas of a rectangular, triangular, trapezoidal or parabolic form, corresponding to a certain power law. In these cases the moments shall be designated by M, i.e., symbols M_{Xm} will be used

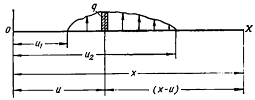

Fig. 1.6

for determining the m-th order moment of a distributed continuous load located to the left of point X with an abscissa x, with respect to this point (Fig. 1.6)

$$M_{Xm} = \int\limits_{u=u_1}^{u=u_2} (x-u)^m \, dF = \int\limits_{u=u_1}^{u=u_2} q\,(x-u)^m \, dx \tag{1.14}$$

If the area corresponds not to the load, but to the cross section of a beam, then its moment is designated by I, and the moment of the m-th order of area F (Fig. 1.1) with respect, say, to axis z-z, will be determined by formula

$$I_{zm} = \int\limits_F y^m \, dF \tag{1.15}$$

where the ordinate y must be taken with its own sign.

If the absolute magnitude of the ordinate is taken, then for the m-th order moment of area F we use the symbol $\overline{I}$; in this case relative to the axis z-z it is determined by the formula

$$\overline{I}_{zm} = \int\limits_F |y|^m \, dF \tag{1.16}$$

The accepted symbols M_{Xm}, I_{zm} and $\overline{I}_{zm}$ are used to designate moments throughout this book.

The term "moment of inertia of an area" should be rejected, for it does not correspond to what it implies. Aragon, for example, also felt the need for another term and he used "moment of the second order". If a special term for this moment is required, then "strength moment" or "strength modulus" seem most appropriate. The term "moment of inertia" should be related only to various moments of the second order. For this reason the quantities T_x and T_y in formulas

$$T_x = \int_F y\rho^2\, dF = \int_F y^3\, dF + \int_F yx^2\, dF \tag{1.17}$$

and

$$T_y = \int_F x\rho^2\, dF = \int_F x^3\, dF + \int_F xy^2\, dF \tag{1.18}$$

should be called "the polar-axial moments of the third order". The term "polar-axial moments of inertia", or, in other words, "polar-axial moments of the second order" should refer only to the following integrals

$$\int_F x\rho\, dF \quad \text{and} \quad \int_F y\rho\, dF$$

In using the term "moments of the third order of areas" the sequence of words should not be changed; for example, it would be wrong to say "moments of areas of the third order", since this expression changes the meaning of the term.

Besides the above mentioned moments of higher order, where the order of a moment was expressed by a positive number, in engineering designing (considering creep, for example), moments of fractional and negative order may be employed.

The following expressions are examples of moments of fractional order:

$$I_{z1/n} = \int_F y^{1/n}\, dF \tag{1.19}$$

$$I_{zm+1/n} = \int_F y^{m+1/n}\, dF \tag{1.20}$$

As for the moments of negative order, their application will be illustrated later.

It is useful to introduce a concept of a positive fractional moment, for example*

$$I_{z\frac{n+1}{n}} = \int_F |y|^{\frac{n+1}{n}}\, dF \tag{1.21}$$

---

* Here $|y|$ is the absolute magnitude of the variable distance of the area dF from the moment axis, i.e., $|y| = y_1$ or $|y| = y_2$.

A sectorial static moment $\int_F \omega \, dF$, a sectorial moment of inertia $\int_F \omega^2 \, dF$ and similar moments of area are widely employed in designing thin-walled constructions.

All the above mentioned moments of various orders (ordinary moments, sectorial moments, moments of fractional orders) belong to the group of geometrical moments, or geometrical quantities. However, for up-to-date designing this mathematical apparatus is not sufficient. In addition to the moments which are considered to be purely geometrical factors, there are other moments which can be called physical-geometric moments, such as kinetic sectorial moments, static moments and moments of inertia dealing with materials of variable "density"; "thermoelastic" characteristics, i.e., "thermoelastic" area, static moments of the "thermoelastic" rigidity of a section, and similarly determined moments of inertia of a "thermoelastic" area.

If a beam with one axis of symmetry is deflected in the plane containing this axis and is subjected to high temperature distributed unevenly across its section and to loads exceeding the elastic limit, we may expect the elastic and plastic characteristics to vary across the section and along the length of the beam. In this case the stress is determined according to the following formula

$$\sigma_x = E_T [1 - \omega (x)] \{[A + By] - \alpha T\} \tag{1.22}$$

where

$$A = \frac{N_x}{EF^{(red)}} + \alpha \frac{S_T^{(red)}}{F^{(red)}} \tag{1.23}$$

$$B = \frac{M_z}{EI_z^{(red)}} + \alpha \frac{I_{zT}}{I_z^{(red)}} \tag{1.24}$$

$$F^{(red)} = \int_F \frac{E_T}{E} [1 - \omega (\varepsilon_x)] \, dF \tag{1.25}$$

$$S_z^{(red)} = \int_F \frac{E_T}{E} [1 - \omega (\varepsilon_x)] \, y \, dF \tag{1.26}$$

$$I_z^{(red)} = \int_F \frac{E_T}{E} [1 - \omega (\varepsilon_x)] \, y^2 \, dF \tag{1.27}$$

$$S_{z_1T}^{(red)} = \int_F \frac{E_T}{E} [1 - \omega (\varepsilon_x)] \, T \, dF \tag{1.28}$$

$$I_{z_1T}^{(red)} = \int_F \frac{E_T}{E} [1 - \omega (\varepsilon_x)] \, Ty \, dF \tag{1.29}$$

are thermoelastic and plastic physical-geometric moments.

Since these moments, as well as the function $\omega(\varepsilon_x)$, depend not only on the local temperature, but also on the unknown stress, the author suggests using the method of successive thermoelastic and plastic approximations, assuming in the first approximation that $\omega(\varepsilon_x) = 0$, i.e., that the deformation is elastic. Under this condition the values of stress and deformation can be determined and the thermoelastic and plastic characteristics calculated and used for repeated calculations.

Thus, from the above mentioned formulas we may conclude that *if the function* $\omega(\varepsilon_x)$, *which is not equal to zero beyond the elastic limit, is presented by an algebraic polynomial with a few members, then the thermoelastic and plastic characteristics will be expressed in terms of physical-geometric moments which may be called thermoelastic and plastic moments of area, or thermo-geometric moments of various orders of area F of the figure.*

The biaxial moments named co-moments were introduced into calculating practice by A. Umansky to determine deflections in beams of continuously varying cross sections. To calculate these co-moments, we must find biaxial moments of various orders according to the flexibility diagram of the beam, which is inverse to rigidity.

A specific feature of the co-moments is that the order of the moment relative to one of the axes varies, whereas the order of the moment relative to the other axis is assumed to be equal to unity; thus,

$$I_{y_1^m y_2} = \int_F z_1^m z_2 \, dF \tag{1.30}$$

where the axes y_1 and y_2 are parallel.

Thus, the moments (uniaxial, biaxial, polar, polar-axial, fractional, positive, sectorial and others) may be determined not only for areas of geometric figures (loads, cross sections, rigidity and flexibility diagrams, non-linear stress-strain diagrams), but also for the variable density of materials, for temperature which is a function of the location of the point, and for time.

2. MOMENTS AS GEOMETRIC CHARACTERISTICS OF BEAM CROSS SECTIONS

In the simplest calculations of beams based on the Bernoulli-Navier hypothesis and Hooke's law, the formulas used deal only with elementary geometric characteristics, such as areas, static moments and moments of inertia of the cross sections, i.e., moments of zero, first and second orders.

In more complicated and precise calculations, as well as in designing beams made of materials not obeying Hooke's law, beams in elasto-plastic state, in creep design and in other cases, the formulas deal with more complicated geometric characteristics, such as: axial, multiaxial, polar, polar-axial, and other geometric and physical-geometric moments of higher order.

For example, in analysing the torsion of prismatic beams in order to obtain an exact solution on the basis of the Saint-Venant theory by means of stress function, two conditions must be fulfilled: (1) the stress function ψ must be equal to zero along the contour of the cross section, and (2) the Laplacian of the second order at any point of the cross section must be constant, i.e., it should not depend on the coordinates z and y of a point in the plane of the section. In this case the Laplacian is equal to

$$\Delta^2\psi = \frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial y^2} = 2G\varphi_0 \qquad (2.1)$$

where G is the shear modulus and φ_0 is the unit angle of torsion of the beam.

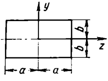

Fig. 2.1

To maintain the above mentioned conditions when computing the torsion of prismatic beams with polygonal (or any other) cross sections, the stress function is taken in the form of a product of contour line equations and a polynomial, in terms of the coordinates. For example, in the case of a beam of a rectangular section (Fig. 2.1), according to the Ritz-Timoshenko method, the following expression is taken for a stress function

$$\psi = (z^2 - a^2)(y^2 - b^2)\sum\sum a_{mn} z^m y^n \qquad (2.2)$$

In this case the first approximation is obtained by taking $a_{00} = k$, while the rest of the coefficients in the polynomial are equal to zero, i.e., when

$$\psi = k\,(z^2 - a^2)\,(y^2 - b^2) \qquad (2.3)$$

However, since the condition expressed by the formula $\Delta^2\psi = \mathrm{const}$ is no longer satisfied, the unit angle of torsion is determined from the condition that the work of the external forces (per unit length) is equal to that of the internal forces

$$\frac{M\varphi_0}{2} - \frac{1}{2G}\int\limits_F (\tau_{xy}^2 + \tau_{xz}^2)\,dF = 0$$

which leads to

$$\varphi_0 = \frac{1}{GM}\int\limits_F (\tau_{xy}^2 + \tau_{xz}^2)\,dF \qquad (2.4)$$

The coefficient k may be determined from the following equation

$$M + 2\int\limits_F \psi\,dF = 0 \qquad (2.5)$$

After substituting (2.3) into (2.5) we obtain

$$M + 2k \int_F (z^2 - a^2)(y^2 - b^2)\, dF = 0 \tag{2.6}$$

The integrals $\int_F z^2\, dF$ and $\int_F y^2\, dF$ in these formulas are uniaxial moments of the second order, and the integral $\int_F z^2 y^2\, dF$ is a biaxial moment of the fourth order

$$\int_F z^2 y^2\, dF = I_{y^2 z^2} \tag{2.7}$$

After appropriate calculations we find

$$k = -\frac{9}{32} \cdot \frac{M}{a^3 b^3} \tag{2.8}$$

and the components of the total tangential stress are

$$\tau_{xy} = \frac{\partial \varphi}{\partial z} = -\frac{9}{16} \cdot \frac{M}{a^3 b^3} (y^2 - b^2)\, z \tag{2.9}$$

and

$$\tau_{xz} = -\frac{\partial \varphi}{\partial y} = \frac{9}{16} \cdot \frac{M}{a^3 b^3} (z^2 - a^2)\, y \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.4), we obtain

$$\varphi_0 = \frac{1}{GM} \int_F \left(\frac{9}{16} \cdot \frac{M}{a^3 b^3} \right)^2 [z^2 (y^2 - b^2)^2 + y^2 (z^2 - a^2)^2]\, dF \tag{2.11}$$

The right side of the formula contains moments $\int_F y^2 z^4\, dF$, $\int_F z^2 y^4\, dF$ and $\int_F z^2 y^2\, dF$, which are particular cases of biaxial moments of higher order.

If we now take the first three terms of (2.2) for a better approximation, we arrive at the following expression for the stress function

$$\varphi = (z^2 - a^2)(y^2 - b^2)\, [k_0 + k_1 (z^2 + y^2)] \tag{2.12}$$

which leads to the calculation of biaxial moments of still higher orders when the above mentioned procedure is carried out.

When using such formulas, it is advisable to calculate the biaxial moments beforehand. For example, a biaxial moment of a rectangular cross section is given by the general expression

$$I_{y^n z^m} = \int_F z^n y^m \, dF = \int_{-a}^{+a} z^n \, dz \int_{-b}^{+b} y^m \, dy = 4\frac{a^{n+1}b^{m+1}}{(n+1)(m+1)} \qquad (2.13)$$

where $2a$ is the height and $2b$ is the base of the rectangle*.

Similar, though more complicated moments of higher order are used for calculating creep in beams of rectangular cross section.

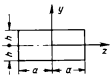

Fig. 2.2

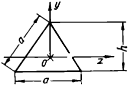

Fig. 2.3

For example, the value I_2 for a rectangle with measurements $2a$ and $2h$ (Fig. 2.2) is expressed by the formula

$$I_2 = 4^{\frac{n+1}{2}+1} \int_0^h \int_0^b [z^2(y^2-h^2)^2 + y^2(z^2-a^2)^2]^{\frac{n+1}{2}} \, dz \, dy \qquad (2.14)$$

Assuming $\frac{n+1}{2} = m$, the integrand can be expanded according to Newton's binomial theorem, and then the moments are calculated by formulas similar to (2.13).

In a similar way, when calculating the torsion of a prismatic beam having an equilateral triangular section with side a (Fig. 2.3), the product of the contour line equations gives an exact solution for the stress function, i.e., a solution satisfying formula (2.1)

$$\psi = k\left(y+\frac{h}{3}\right)\left[\left(y-\frac{2}{3}h\right)^2 - 3z^2\right] \qquad (2.15)$$

In this case, according to (2.1),

$$\Delta^2\psi = -4kh = -2G\varphi_0$$

hence

$$\varphi_0 = \frac{2kh}{G} \qquad (2.16)$$

To determine the value of the coefficient k we can use the condition that the sum of the moments of the internal and external forces relative to the beam axis is equal to zero, i.e.,

---

* Under the condition that both indices (m and n) are even numbers; otherwise the biaxial moment will be equal to zero.

$$M+\int_F (\tau_{xz}y-\tau_{xy}z)\,dF=0 \tag{2.17}$$

where

$$\tau_{xz}=-\frac{\partial\psi}{\partial y};\quad \tau_{xy}=\frac{\partial\psi}{\partial z}$$

Employing formula (2.15), the equilibrium equation may be reduced to the following form:

$$M+k\int_F(-3y^2+2hy+3z^2)\,y\,dF+k\int_F(6zy+2hz)\,z\,dF=0 \tag{2.18}$$

This formula contains the following moments of a triangular area

$$I_{z^2}=\int_F y^2\,dF;\quad I_{zy^2}=\int_F z^2y\,dF;\quad I_{z^3}=\int_F y^3\,dF \tag{2.19}$$

The solution obtained shows that it is advisable to calculate beforehand biaxial moments of cross-sectional areas of beams in torsion, as it was done in (2.13) for a rectangular section and in (2.19) for a triangular section. Similar calculations may be useful when calculating rod deflections. For example, in the case of a beam subjected to bending accompanied by torsion it may be useful to determine moments of higher order, such as I_{z^2y} and I_{zy^2}, by calculating corresponding integrals.

Normal stresses in a deflected beam of large curvature are determined by the formula

$$\sigma=\frac{N}{F}+\frac{M}{\rho F}-\frac{M}{I'}\;\frac{y}{1-\frac{y}{\rho}} \tag{2.20}$$

where y is the distance from the centre axis*.

The equivalent moment of inertia I' in (2.20) may be calculated by the formula

$$I'=\int_F\frac{y^2\,dF}{1-\frac{y}{\rho}}=\int_F y^2\left(1+\frac{y}{\rho}+\frac{y^2}{\rho^2}+\dots\right)dF=$$

$$=I+\frac{1}{\rho}\int_F y^3\,dF+\frac{1}{\rho^2}\int_F y^4\,dF+\dots \tag{2.21}$$

---

* **The author of this book proved that in designing curved bars, which are widely used in mechanical engineering, formula (2.20) may be used if the equivalent moment I' is replaced by the ordinary moment I.**

Members of this rapidly converging series are uniaxial moments of higher order of the cross-sectional area of the beam with respect to the centre axis.

For the design of thin-walled elastic rods the following "new" geometric characteristics were introduced by V. Vlassov

$$\left.\begin{aligned} U_z &= \int_F y\rho^2\,dF \\ U_y &= \int_F z\rho^2\,dF \end{aligned}\right\} \tag{2.22}$$

As already mentioned, these are polar-axial moments of the third order (Fig. 2.4). Moments of this type may be expanded into uniaxial

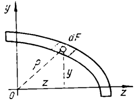

Fig. 2.4

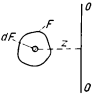

Fig. 2.5

and biaxial moments by the following formulas

$$\left.\begin{aligned} U_z &= \int_F y^3\,dF + \int_F z^2y\,dF = I_{z^3} + I_{zy^2} \\ U_y &= \int_F z^3\,dF + \int_F zy^2\,dF = I_{y^3} + I_{z^2y} \end{aligned}\right\} \tag{2.23}$$

Let us consider the application of moments to the analysis of bodies of revolution. The moment of the n-th order of the mass of a body (Fig. 2.5) formed by a plane figure F rotating around axis O-O is expressed by

$$I_{O^n}(m) = \int \frac{\gamma}{g}\cdot 2\pi z\cdot dF\cdot z^n = \frac{\gamma}{g}\,2\pi I_{O^{n+1}}(F) \tag{2.24}$$

where $I_{O^n}(m)$ is the moment of the n-th order of the mass, $I_{O^{n+1}}(F)$ is the moment of order $(n + 1)$ of area F, γ is the weight of a unit volume.

Thus, the moment of any order n of the mass of a body of revolution with respect to the axis of revolution can be expressed in terms of the moment of order $(n + 1)$ of the area of the revolving figure. When $n = 0$, we have the well-known Guldin's formula for determining the volume of the body of revolution. When $n = 2$, a formula is derived for determining the moment of inertia of the mass of the body of revolution*.

$$I_{O^2}(m) = I_m = \frac{\gamma}{g} 2\pi I_3 \tag{2.25}$$

where I_3 is the moment of the third order of area F with respect to axis O-O.

For a particular case, when F is a rectangle (Fig. 2.6), its revolution around axis O-O forms a hollow cylinder with a moment of inertia of its mass equal to

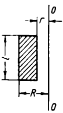

Fig. 2.6

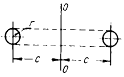

Fig. 2.7

$$I(m) = 2\pi \frac{\gamma}{g} \cdot \frac{R^4 - r^4}{4}\, l \tag{2.26}$$

Similarly, a tore is formed by a circle rotating around the O-O axis (Fig. 2.7); the moment of inertia of its mass is equal to

$$I(m) = 2\pi \frac{\gamma}{g}\left(3c\frac{\pi r^4}{4} + c^3 \pi r^2\right) \tag{2.27}$$

Finally, when studying the geometrical sections of beams, moments of higher order may be used to calculate the deflection of the construction elements made of brittle materials, which do not obey Hooke's law, or of plastic materials beyond the elastic limit.

Among possible stress-strain relationships those of the parabolic-type deserve the most attention because they allow to develop a rather simple theory for designing beams subjected to bending. For this purpose, geometric properties of the cross sections should be analysed thoroughly on the basis of the theory of higher moments. It means that besides the well-known static moments and moments of inertia of plane geometric figures, moments of higher order must be investigated by deriving corresponding formulas and developing theoretical principles beforehand.

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* Formula (2.25) was advanced by Yates in a paper "Third Moments" in "Journal of Applied Mechanics", No. 2, 1936.

Designing Beams Not Subject to Hooke's Law

According to Prandtl, in designing parts made of plastic materials the usual tension diagram (Fig. 2.8) can be replaced by a broken-line diagram (Fig. 2.9); the strain hardening which follows after yielding is disregarded. The first part of the diagram corresponds to the elastic state of material, and the second part—to the plastic state with an infinitely long flow section.

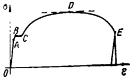

Fig. 2.8

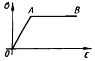

Fig. 2.9

The second part of the diagram, AB, represents the state of material only to a certain degree of approximation. Several scientists tried to introduce other σ-ε diagrams which would represent the behaviour of material beyond the elasticity limit. For example, F. Stephan suggested a diagram (Fig. 2.10) for plastic materials consisting of three sections, the third (hardening) section being a 10/3-power parabola.

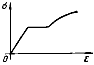

Fig. 2.10

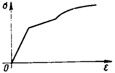

Fig. 2.11

J. Fritsche and K. Jezek suggested a diagram similar to Stephan's, the second section of Jezek's diagram being a slightly sloped straight line (Fig. 2.11).

N. Bezookhov, on the contrary, replaced the curved part of the diagram by a broken line. Although this idea is considered valuable, it must be pointed out that even the much simpler Prandtl's diagram (Fig. 2.9), when used for analysing the deflections of a statically determinate beam under a simple loading, leads to quite complicated calculations. A. Ilyushin's "method of elastic solutions" and the "method of elasto-plastic solutions" do not make the problem much easier. Neither do Bezookhov's nor Bernstein's methods. Difficulties arise from the elasto-plastic state of the beam and from the necessity of employing two different relationships within one stress-strain diagram. These difficulties are not diminished when three or more relationships have to be used instead of two.

That is why, besides moments of higher order, it is advisable to employ functional discontinuators introduced by N. Gersevanov in order to combine several functions representing the Clayperon diagram into one generalized mathematical expression. Corresponding formulas will contain moments as well as discontinuators or any other notation for integrals of discontinuous functions similar to Stieltjes integrals.

In order to represent all the possible states of the material by one mathematical law, or, in other words, to replace the diagram, composed of different phases, by one approximate expression, the following algebraic function may be introduced

$$\sigma = a_1\varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \ldots \tag{2.28}$$

This expression is not at variance with the general relationship between σ and ε

$$\sigma = f\ (\varepsilon) \tag{2.29}$$

After expanding this function into MacLaurin's series the following expression is obtained

$$\sigma = f(\varepsilon) = f(0) = \frac{\varepsilon}{1} f'(0) + \frac{\varepsilon^2}{1\cdot 2} f''(0) + \ldots \tag{2.30}$$

When a combination of a straight line with parabolas of different powers is selected to represent $f(\varepsilon)$, it must be kept in mind that parabolas of odd power have points of inflection; this may be useful when approximating the tension-compression diagram at these points.

For brittle materials, several functions were suggested for the relationship between σ and ε. Among these functions the most popular is the power law

$$\varepsilon = \alpha\sigma^m \tag{2.31}$$

or

$$\sigma = C\varepsilon^n \tag{2.32}$$

According to N. Mitinsky, the power law was introduced long before C. Bach by Bülfinger, who suggested the following formulas

$$\varepsilon_z = \alpha_1\sigma_z^m \quad \text{and} \quad \varepsilon_d = \alpha_2\sigma_d^n$$

for the tensioned and the compressed zones of the bent beam, respectively.

Differentiation of (2.31) leads to

$$\frac{d\sigma}{d\varepsilon}=\frac{1}{\alpha m}\cdot\frac{1}{\sigma^{m-1}} \tag{2.33}$$

However, when m slightly exceeds unity (as in tests carried out by Bach, Schulle and Pinegin) the dimensions of ε do not satisfy formula (2.31) and as the stress σ approaches zero, the derivative $d\sigma/d\varepsilon$ in (2.33) tends to infinity.

Föppl suggested the following formula

$$\varepsilon=\alpha\left(\frac{\sigma}{\sigma_0}\right)^m \tag{2.34}$$

which means the same as (2.31) if σ_0 is assumed equal to unity.

Since it is impossible to accept that as $\sigma = 0$ the variable modulus of elasticity $d\sigma/d\varepsilon \to \infty$, many researchers tried to derive other functions. Formulas for brittle materials following from (2.28) and (2.30) were suggested especially often.

If only the first term is left in (2.28), we arrive at Hooke's law, while two or three terms give new relationships of the parabolic type. These functions agree with experimental results for certain materials.

V. Manzhalovsky employed Emperher's method and proposed the following two-term formulas of the parabolic type on the basis of the tests carried out by Ja. Stolyarov and B. Skramtayev

$$\sigma_1 = \alpha_1\varepsilon - \beta_1\varepsilon^2 \quad \text{(for compression)} \tag{2.35}$$

and

$$\sigma_2 = \alpha_2\varepsilon - \beta_2\varepsilon^2 \quad \text{(for tension)} \tag{2.36}$$

where α_1, α_2, β_1 and β_2 are empiric elasticity constants. In his studies Manzhalovsky used additional conditions: (1) when $\varepsilon = 0$, the moduli of elasticity are equal ($E_1 = E_2$); (2) when $\varepsilon_2 = \max \varepsilon_2$, the modulus $E_2 = 0$ (i.e., at the point of maximum strain the tangent to Clayperon's curve is horizontal).

Trinomial formulas of the parabolic type were suggested for cast iron by Hodgkinson in 1822 and by Cox in 1850. Later, L. Tyte proposed to use a similar relationship for metal wire

$$\varepsilon = a\sigma + b\sigma^2 + c\sigma^3 \tag{2.37}$$

while Broude did the same for concrete

$$\begin{aligned}\sigma_{comp} = 2400 - 10^9\,[0.854\,(1.4\cdot10^{-3} - \varepsilon)^2 +\\ + 2040\,(1.4\cdot10^{-3} - \varepsilon)^3]\end{aligned} \tag{2.38}$$

This expression may be transformed to the following form

$$\sigma_{comp} = \alpha + \beta\varepsilon + \gamma\varepsilon^2 + \delta\varepsilon^3 \tag{2.39}$$

Besides the exponential and power laws, hyperbolic relationships were also suggested for brittle materials. For example, Cox introduced the following function for cast iron

$$\varepsilon = \frac{\sigma}{a - b\sigma} \tag{2.40}$$

Later, Grubler, Schlechtweg and Deitler developed a theory for bars and discs made of brittle materials; they employed the following hyperbolic laws

$$\sigma = \frac{\varepsilon}{A + B\varepsilon} \tag{2.41}$$

and

$$\varepsilon = \frac{\sigma}{E} \; \frac{1}{1 - \alpha\sigma + \beta\sigma\sqrt{\sigma}} \tag{2.42}$$

However, calculations carried out by these researchers for extremely simple members were so complicated, that it is highly unlikely that this method will be used for more complicated problems in the future. On the other hand, any hyperbolic relationship between σ and ε may be approximated by a parabolic function; indeed, by expanding the hyperbolic formula into an infinite series, we obtain a rapidly converging series when $\beta\varepsilon < 1$

$$\sigma = \frac{\alpha\varepsilon}{1 + \beta\varepsilon} = \alpha\varepsilon\,[1 - \beta\varepsilon + (\beta\varepsilon)^2 - \dots] \tag{2.43}$$

the first two or three terms of which are sufficient for quite an exact approximation.

Besides the above mentioned equations, L. Onishchik proposed a logarithmic curve for large-block masonry and concrete massives

$$\varepsilon = \frac{1}{E_0} \int_0^\sigma \frac{d\sigma}{1 - \left(\frac{\sigma}{R}\right)^k} \tag{2.44}$$

where

$$E = \frac{d\sigma}{d\varepsilon} = E_0 \left[1 - \left(\frac{\sigma}{R}\right)^k \right] \tag{2.45}$$

Formulas (2.44) and (2.45) contain the initial modulus of elasticity E_0 and exponent k as parameters. Onishchik suggested a simplified formula which does not correlate so well with experimental data. This formula contains a variable modulus of elasticity determined as follows

$$E = E_0 \left(1 - \frac{\sigma}{R'}\right) \tag{2.46}$$

where $R' = 1.10R$.

The most accurate formula was proposed by Onishchik; this was proved by experiments and by comparing the results with formulas introduced by Bach and Schulle (power type) $\varepsilon = \alpha\sigma^m$, by Schreider (hyperbolic type) $\varepsilon = \dfrac{\sigma}{E\left(1-\dfrac{\sigma}{1.1R}\right)}$ and with a parabolic formula $\sigma = \alpha\varepsilon - \beta\varepsilon^2$.

In some particular cases a binomial parabolic formula may be used; deviations from the exact solution did not exceed 27% even in the most unfavourable problems solved by such formulas. The extreme deviations will evidently be lower in case a trinomial parabolic function is employed. Still better results will be obtained if binomial formulas of the form $\sigma = \alpha\varepsilon^2 + \beta\varepsilon^3$ or $\sigma = \gamma_1\varepsilon + \gamma_3\varepsilon^3$ are used.

As to arbitrary "graphical" relationships between σ and ε, i.e., laws without any analithic form, their application is not advisable because the results obtained are usually reliable only to a small degree and can be checked only by new graphical calculations.

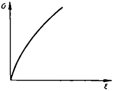

Fig. 2.12

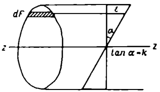

Fig. 2.13

Let us consider an n-nomial parabolic stress-strain relationship (Fig. 2.12) for a beam in pure flexure:

$$\sigma = a_1\varepsilon + \frac{a_2}{2}\varepsilon^2 + \frac{a_3}{3!}\varepsilon^3 + \ldots + \frac{a_n\varepsilon^n}{n!} = \sum_{i=1}^{i=n}\frac{a_i\varepsilon^i}{i!} \qquad (2.47)$$

Assuming Bernulli's hypothesis concerning plane sections ($\varepsilon = ky$), we arrive at $dP = \sigma\, dF$, and from statical equations we find

$$\sum_{i=1}^{i=n}\frac{a_ik^i}{i!}\int_F y^i\, dF = 0 \qquad (2.48)$$

and

$$M_z = \sum_{i=1}^{i=n}\frac{a_ik^i}{i!}\int_F y^{i+1}\, dF \qquad (2.49)$$

The right-hand side integrals of the formulas are moments of corresponding orders of area F with respect to axis z-z (Fig. 2.13); therefore equations (2.48) and (2.49) may be rewritten as

$$\sum_{i=1}^{i=n} \frac{a_i k^i}{i!} I_i = 0 \tag{2.50}$$

and

$$\sum_{i=1}^{i=n} \frac{a_i k^i}{i!} I_{i+1} = M_z \tag{2.51}$$

Thus, strength and rigidity calculations of brittle materials may be based on a parabolic stress-strain relationship and on a full analysis of geometrical properties of the cross sections by means of the theory of moments of higher order. Besides the well-known statical moments and moments of inertia of plane geometric figures, moments of the third and higher orders should be employed as well as moments of fractional orders (for creep design, in particular) and more complicated concepts, such as physical and geometrical moments. Being applied to Hooke's law, the application of cross-sectional "geometry" led to a number of rather refined constructions, such as the well-known ellipse of inertia, Mohr's and Zeiner's circles. It is difficult to foresee all the possible ideas concerning the complex "geometry" of sections but Culman's and Nels's graphical methods of determining moments of inertia are also easily applied to the moments of higher order.

Application of a single parabolic relationship as a characteristic of the tension-compression diagrams of plastic materials is a much more complicated problem than selection of coefficients in the parabolas representing brittle materials. Furthermore, the tension-compression diagram cannot be mechanically applied to beams in flexure; several additional factors must be taken into account, such as the "unloading" parameters, creep*, temperature, etc.

Although distribution of stress across the section does not indicate the strength of a bar or a system of bars, nevertheless the idea of approximating different parts of a tension-compression diagram with a single rational function deserves consideration. Such a function will make it possible to develop a theory of design on the basis of moments of higher order.

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* The creep design theory employs moments of fractional orders. Stabilized creep design is simpler than many other problems beyond the Hooke's law, because it is based on a general monomial stress-strain relationship.

Rotation of Axes and Associated Changes in Moments of the Third Order of a Plane Figure

Consider the third moments of a plane figure F with respect to the initial axes z and y

$$\left.\begin{aligned} I_{z^3} &= \int_F y^3\,dF, & I_{y^3} &= \int_F z^3\,dF \\ I_{z^2y} &= \int_F zy^2\,dF, & I_{zy^2} &= \int_F z^2y\,dF \end{aligned}\right\} \qquad (2.52)$$

as well as all the moments of lower orders.

Let us determine the moments of the third order of area F with respect to axes u and v, which are turned through a positive angle α with respect to the initial axes, i.e.,

$$\left.\begin{aligned} I_{u^3} &= \int_F v^3\,dF, & I_{v^3} &= \int_F u^3\,dF \\ I_{u^2v} &= \int_F uv^2\,dF, & I_{uv^2} &= \int_F u^2v\,dF \end{aligned}\right\} \qquad (2.53)$$

Since

$$u = z\cos\alpha + y\sin\alpha$$

and

$$v = y\cos\alpha - z\sin\alpha$$

then

$$\begin{aligned} I_{u^3} &= \int_F v^3\,dF = \int_F (y\cos\alpha - z\sin\alpha)^2\,dF = \cos^3\alpha\int_F y^3\,dF - \\ &- 3\cos^2\alpha\sin\alpha\int_F y^2z\,dF + 3\cos\alpha\sin^2\alpha\int_F yz^2\,dF - \\ &- \sin^3\alpha\int_F z^3\,dF = I_{z^3}\cos^3\alpha - I_{z^2y}\cdot 3\sin\alpha\cos^2\alpha + \\ &+ I_{zy^2}\cdot 3\sin^2\alpha\cos\alpha - I_{y^3}\sin^3\alpha \end{aligned} \qquad (2.54)$$

The other third moments are determined similarly

$$\begin{aligned} I_{v^3} &= \int_F u^3\,dF = \int_F (z\cos\alpha + y\sin\alpha)^3\,dF = I_{y^3}\cos^3\alpha + \\ &+ I_{zy^2}\cdot 3\sin\alpha\cos^2\alpha + I_{z^2y}\cdot 3\sin^2\alpha\cos\alpha + I_{z^3}\sin^3\alpha \end{aligned} \qquad (2.55)$$

$$I_{u^2v}=\int_F uv^2 dF=\int_F (z\cos\alpha+y\sin\alpha)(y\cos\alpha-z\sin\alpha)^2 dF=$$
$$=I_{z^3}\sin\alpha\cos^2\alpha-I_{z^2y}\cos\alpha(2\sin^2\alpha-\cos^2\alpha)-$$
$$-I_{zy^2}\sin\alpha(2\cos^2\alpha-\sin^2\alpha)+I_{y^3}\sin^2\alpha\cos\alpha \tag{2.56}$$
$$I_{uv^2}=\int_F u^2v\, dF=\int_F (z\cos\alpha+y\sin\alpha)^2(y\cos\alpha-z\sin\alpha)\, dF=$$
$$=I_{z^3}\sin^2\alpha\cos\alpha-I_{zy^2}\cos\alpha(2\sin^2\alpha-\cos^2\alpha)+$$
$$+I_{z^2y}\sin\alpha(2\cos^2\alpha-\sin^2\alpha)-I_{y^3}\sin\alpha\cos^2\alpha \tag{2.57}$$

The formulas obtained may be used to establish the relationship between biaxial and uniaxial moments of the third order.

After multiplying both parts of (2.54) by sin α and both parts of (2.55) by cos α and adding the results term-by-term, the following equation is obtained

$$I_{u^3}\sin\alpha+I_{v^3}\cos\alpha=\frac{1}{2}I_{z^3}\sin 2\alpha+I_{y^3}\cos 2\alpha+\frac{3}{2}I_{zy^2}\sin 2\alpha \tag{2.58}$$

Now, after multiplying both parts of (2.55) by sin α and both parts of (2.54) by cos α and subtracting the second from the first term-by-term, another equation is obtained

$$I_{v^3}\sin\alpha-I_{u^3}\cos\alpha=\frac{1}{2}I_{y^3}\sin 2\alpha-I_{z^3}\cos 2\alpha+\frac{3}{2}I_{z^2y}\sin 2\alpha \tag{2.59}$$

The last two equations give the following relationships

$$I_{zy^2}=\frac{2}{3}\cdot\frac{I_{v^3}\cos\alpha+I_{u^3}\sin\alpha-\frac{1}{2}I_{z^3}\sin 2\alpha-I_{y^3}\cos 2\alpha}{\sin 2\alpha} \tag{2.60}$$

and

$$I_{z^2y}=\frac{2}{3}\cdot\frac{I_{v^3}\sin\alpha-I_{u^3}\cos\alpha+I_{z^3}\cos 2\alpha-\frac{1}{2}I_{y^3}\sin 2\alpha}{\sin 2\alpha} \tag{2.61}$$

This solution may be simplified when $\alpha = 45°$; then formulas (2.60) and (2.61) take the form

$$I_{zy^2}=\frac{\sqrt{2}}{3}\left(I_{u^3}+I_{v^3}-\frac{\sqrt{2}}{2}I_{z^3}\right) \tag{2.62}$$

and

$$I_{z^2y}=\frac{\sqrt{2}}{3}\left(I_{v^3}-I_{u^3}-\frac{\sqrt{2}}{2}I_{y^3}\right) \tag{2.63}$$

The last two formulas are the relationships between biaxial and uniaxial moments of the third order. They are similar to the formulas for axial and centrifugal moments of inertia (i.e., moments of second order), which are well known in the theory of resistance of materials.

Since the new coordinate system (u, v) is turned with respect to the initial one (z, y) through a positive angle α, then $u = z\cos\alpha + y\sin\alpha$ and $v = y\cos\alpha - z\sin\alpha$. Therefore,

$$\frac{\partial v}{\partial\alpha} = -u, \quad \frac{\partial u}{\partial\alpha} = v \tag{2.64}$$

Then

$$I_{u^3} = \int\limits_F v^3\,dF = \int\limits_F (y\cos\alpha - z\sin\alpha)^3\,dF = \\ = I_{z^3}\cos^3\alpha - 3I_{z^2y}\sin\alpha\cos^2\alpha + 3I_{zy^2}\sin^2\alpha\cos\alpha - I_{y^3}\sin^3\alpha \tag{2.65}$$

$$I_{v^3} = \int\limits_F u^3\,dF = \int\limits_F (z\cos\alpha + y\sin\alpha)^3\,dF = I_{y^3}\cos^3\alpha + \\ + 3I_{y^2z}\cos^2\alpha\sin\alpha + 3I_{yz^2}\cos\alpha\sin^2\alpha + I_{z^3}\sin^3\alpha \tag{2.66}$$

By differentiating (2.65) and (2.66) with respect to α, the following formulas are obtained

$$\frac{\partial I_{u^3}}{\partial\alpha} = \int\limits_F 3v^2\frac{\partial v}{\partial\alpha}\,dF = -3I_{u^2v}$$

and similarly

$$\frac{dI_{v^3}}{d\alpha} = +3I_{uv^2} \tag{2.67}$$

The value I_{u^3} reaches its maximum when $I_{u^2v} = 0$; this means that if v is the axis of symmetry of the figure, then the third moment with respect to axis u will reach its maximum.

The following formulas for biaxial moments of the third order with respect to axes turned through angle α are determined likewise

$$\left.\begin{aligned} I_{u^2v} &= I_{z^3}\sin\alpha\cos\alpha + I_{z^2y}\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) + \\ &\quad + I_{z^2y}\sin\alpha(\sin^2\alpha - 2\cos^2\alpha) + I_{y^3}\sin^2\alpha\cos\alpha \\ I_{uv^2} &= I_{z^3}\sin^2\alpha\cos\alpha + I_{z^2y}\sin\alpha(2\cos^2\alpha - \sin^3\alpha) + \\ &\quad + I_{zy^2}\cos\alpha(\cos^2\alpha - 2\sin^2\alpha) - I_{y^3}\sin\alpha\cos^2\alpha \end{aligned}\right\} \tag{2.68}$$

Differentiation with respect to α leads to

$$\frac{\partial}{\partial\alpha} I_{u^2v} = -2I_{u^2v} + I_{u^3}, \quad \frac{\partial}{\partial\alpha} I_{uv^2} = 2I_{u^2v} - I_{v^3} \tag{2.69}$$

If the axis z is the axis of symmetry of the figure, then $I_{z^3} = 0$, $I_{zy^2} = 0$, $I_{zy} = 0$, whereas I_{y^3} and I_{z^2y} reach their maximum, and by means of (2.53) the maximum of I_{y^3} can be determined; the maximum of I_{u^3} is determined from the following equation when $\alpha = \alpha_0$

$$I_{u^2v} = I_{x^2y} \cos\alpha_0 (\cos^2\alpha_0 - 2\sin^2\alpha_0) + I_{y^3} \sin^2\alpha_0 \cos\alpha_0 = 0$$

which leads to

$$\cos\alpha_0 \left[I_{x^2y}(1 - 3\sin^2\alpha_0) + I_{y^3}\sin^2\alpha_0\right] = 0; \tag{2.70}$$

This formula gives two solutions: either $\cos\alpha_0 = 0$ and $\alpha_0 = \pi/2$ then I_{y^3} has one of the extreme values, or

$$I_{x^2y}(1 - 3\sin^2\alpha_0) + I_{y^3}\sin^2\alpha_0 = 0,$$

which means that

$$\sin\alpha_0 = \pm\sqrt{\frac{I_{x^2y}}{I_{x^2y} - I_{y^3}}} \tag{2.71}$$

$$\cos\alpha_0 = \pm\sqrt{\frac{2I_{x^2y} - I_{y^3}}{3I_{x^2y} - I_{y^3}}} \tag{2.72}$$

$$\tan^2\alpha_0 = \frac{2I_{x^2y}}{2I_{x^2y} - I_{y^3}} \tag{2.73}$$

From (2.70), (2.71), and (2.65) the following expression may be obtained

$$\frac{\text{max}}{\text{min}} I_{u^3} = -3I_{x^2y}\sin^2\alpha_0\cos^2\alpha_0 - I_{y^3}\sin^3\alpha_0 =$$

$$= \pm \frac{2I_{x^2y}^{3/2}}{(3I_{x^2y} - I_{y^3})^{1/2}} \tag{2.74}$$

Formula (2.74) gives the magnitude of the second extremum of the uniaxial moment of the third order. The following formulas clarify which values of I_{u^3} exceed the rest when $\alpha = \alpha_0$ and $\alpha = \pi/2$

$$\frac{2I_{x^2y}}{(3I_{x^2y} - I_{y^3})} > I_{y^3} \tag{2.75}$$

hence

$$4I^3_{x^2y} > (3I_{x^2y} - I^3_{y^3} - I^3_{y^3}) \tag{2.76}$$

In order to clarify all the relationships governing the changes in the uniaxial third moment let us examine its symmetry. Let $\alpha_3 = -\alpha_1$, then $\sin\alpha_3 = -\sin\alpha_1$, $\cos\alpha_3 = -\cos\alpha_1$ and

$$I_{u^3_3} = I_{x^3}\cos^3\alpha_3 - 3I_{x^2y}\sin\alpha_3\cos^2\alpha_3 + 3I_{xy^2}\sin^2\alpha_3\cos\alpha_3 -$$

$$- I_{y^3}\sin^3\alpha_3 = I_{x^3}\cos^3\alpha_1 + 3I_{x^2y}\sin\alpha_1\cos^2\alpha_1 +$$

$$+ 3I_{xy^2}\sin^2\alpha_1\cos\alpha_1 + I_{y^3}\sin^3\alpha_1$$

If an axis of symmetry exists, then

$$I_{u_3^3} = 3I_{z^2y} \sin\alpha_1 \cos^2\alpha_1 + I_{y^3} \sin^3\alpha_1 = -I_{u_3^3} \qquad (2.77)$$

Thus, if z-z is the axis of symmetry, then the moment of the third order will be symmetrical with respect to this axis, the symmetry being of the second type.

Changes in Third Moments Associated with Parallel Translation of Axes

(1) Uniaxial Moments of the Third Order

Let z_0 and y_0 be the central axes of a plane figure, and z and y—the parallel axes, transferred from the central axes through distances b and a, respectively. Then

$$I_{z^3} = \int_F (y_0 + b)^3\, dF = \int_F y_0^3\, dF + 3b \int_F y_0^2\, dF +$$

$$+ 3b^2 \int_F y_0\, dF + b^3F = I_{z_0^3} + 3bI_{z_0^2} + b^3F \qquad (2.78)$$

Similarly,

$$I_{y^3} = I_{y_0^3} + 3aI_{y_0^2} + a^3F \qquad (2.79)$$

(2) Biaxial Moments of the Third Order

$$I_{z^2y} = \int_F (y_0 + b)^2 (z_0 + a)\, dF = I_{z_0^2y_0} + 2bI_{z_0y_0} + aI_{z_0^2} + ab^2F \qquad (2.80)$$

Similarly,

$$I_{zy^2} = I_{z_0y_0^2} + 2aI_{z_0y_0} + bI_{y_0^2} + a^2bF \qquad (2.81)$$

If z_0-z_0 and y_0-y_0 are the principal axes, then $I_{z_0y_0} = 0$; therefore

$$I_{z^2y} = I_{z_0^2y_0} + aI_{z_0^2} + ab^2F \qquad (2.82)$$

$$I_{zy^2} = I_{z_0y_0^2} + bI_{y_0^2} + a^2bF \qquad (2.83)$$

(3) Polar-Axial Moments of the Third Order

$$I_{\rho_0^2z_0} = \int_F \rho_0^2 y_0\, dF = \int_F (z_0^2 + y_0^2)\, y_0\, dF = I_{z_0y_0^2} + I_{z_0^3} \qquad (2.84)$$

$$I_{\rho_0^2y_0} = \int_F \rho_0^2 z_0\, dF = \int_F (z_0^2 + y_0^2)\, z_0\, dF = I_{y_0^3} + I_{z_0^2y_0} \qquad (2.85)$$

$$I_{c^2z}=\int_F \rho^2 y\,dF=I_{zy^2}+I_{z^3}=I_{z_0y_0^2}+2aI_{z_0y_0}+a^2bF+bI_{y_0^2}+$$
$$+I_{z_0^3}+3bI_{z_0^2}+b^3F=I_{z_0^3}+3bI_{z_0^2}I_{z_0y_0^2}2aI_{z_0y_0}+bI_{y_0^2}+$$
$$+b\,(a^3+b^2)\,F \tag{2.86}$$

$$I_{c^2y}=\int_F \rho^2 z\,dF=I_{z^2y}+I_{y^3}=I_{y_0^3}+3aI_{y_0^2}+I_{z_0^2y_0}+$$
$$+2bI_{z_0y_0}+aI_{z_0^2}+a\,(a^2+b^2)\,F \tag{2.87}$$

If the figure has an axis of symmetry (for example, axis y_0) and z_0 and y_0 are the principal axes of inertia, then

$$I_{y_0^3}=0,\ I_{z_0^2y_0}=0 \quad \text{and} \quad I_{z_0y_0}=0$$

Therefore

$$I_{z^3}=I_{z_0^3}+3bI_{z_0^2}+b^3F \tag{2.88}$$

$$I_{y^3}=3aI_{y_0^2}+a^3F \tag{2.89}$$

$$I_{z^2y}=aI_{z_0^2}+ab^2F=b\,(I_{z_0^2}+b^2F)=aI_{z^2} \tag{2.90}$$

$$I_{zy^2}=I_{z_0y_0^2}+bI_{y_0^2}+a^2bF=I_{z_0y_0^2}+b\,(I_{y_0^2}+a^2F)=I_{z_0y_0^2}+bI_{y^2} \tag{2.91}$$

$$I_{c_0^2z_0}=I_{z_0y_0^2}+I_{z_0^3} \tag{2.92}$$

$$I_{c_0^2y_0}=0 \tag{2.93}$$

$$I_{c^2z}=I_{z_0^3}+3bI_{z_0^2}+bI_{y_0^2}+b\,(a^2+b^2)\,F \tag{2.94}$$

and

$$I_{c^2y}=3aI_{y_0^2}+aI_{z_0^2}+a\,(a^2+b^2)\,F \tag{2.95}$$

Axes for Maximum Values of Moments of Even Order

In addition to the above mentioned propositions concerning moments of the third order, we shall present some theorems that deal with the extreme values of moments of any order with respect to a parallel transfer of axes.

Let *O-O* be the central axis of geometrical figure F, axis *1-1* an off-centre axis at a distance a from the centre. Then the moment of an even order ($2n$) with respect to axis *1-1* is

$$I_{1^{2n}}=\int_F z_1^{2n}\,dF=\int_F (z_0-a)^{2n}\,dF \tag{2.96}$$

The integral in this formula contains the parameter a. By changing a, i.e., by transferring the axis to another parallel position, different values of the moment of order $2n$ will be obtained. The extreme values will appear when

$$\frac{\partial I_{1\,2n}}{\partial a}=0, \text{ or } \frac{\partial}{\partial a}\int_F (z_0-a)^{2n}\,dF=\int_F \frac{\partial}{\partial a}(z_0-a)^{2n}\,dF=$$
$$=-2n\int_F (z_0-a)^{2n-1}\,dF=-2nI_{1\,2n-1}=0, \text{ or } I_{1\,2n-1}=0 \quad (2.97)$$

Of all parallel axes the one with an odd moment $I_{1\,2n-1}=0$ has the extreme value of an even moment ($I_{1\,2n}$); such an axis is called the central axis of the odd order $(2n-1)$.

To judge whether the extreme moment $I_{1\,2n}$ is the largest or the smallest possible value, we must examine the second derivative with respect to the parameter and determine its sign

$$\frac{\partial^2}{\partial a^2} I_{1\,2n}=2n(2n-1)\int_F (z-a)^{2n-2}\,dF=2n\,(2n-1)\,I_{1\,2n-2} \quad (2.98)$$

Since the product $2n\,(2n-1)$ is positive, the sign of the derivative will be the same as that of the moment of the order $(2n-2)$; the latter, however, has the sign of area F (that is, if the total area F of the figure is positive, then all even moments of this area with respect to any axis are positive; if area F is negative, then all even moments are negative; negative areas may occur in case figure F is not a cross section or a part of a cross section, but for example, a distributed loading, both uniform or non-uniform).

Thus, it can be stated that among all the moments of an even order $2n$ of a positive geometrical figure the minimum value is associated with the axis, which has a zero moment of an odd order $(2n-1)$.

A well-known particular case of this theorem is the uniaxial moment of the second order (that is, the moment of inertia), which has its minimum with respect to the central axis of the first order, i.e., the axis with respect to which the moment of the first order (statical moment) is equal to zero.

If the total area F is negative, then all the moments of even orders will also be negative, and the moment of even order $2n$ will reach its largest possible value (i.e., a negative value with the smallest possible absolute value) at the central axis of the order $(2n-1)$ with respect to which the moment of odd order $(2n-1)$ is equal to zero.

Extreme Values of a Biaxial Moment

In a similar way we can examine the extreme values of a biaxial moment. Let the following expression be a biaxial moment of $(m+n)$-th order of figure F (i.e., of the m-th order with respect to the central axis *O-O* and of the *n*-th order with respect to the parallel off-centre axis *1-1* at a distance a from the centre

$$I_{0^m1^n} = \int_F z^m (z-a)^n \, dF \tag{2.99}$$

It can be observed by analysing the derivative with respect to a, that a biaxial moment $I_{0^m1^n}$ reaches its extremum with respect to the axis, relative to which the biaxial moment of the preceding order is equal to zero.

Similarly, if both axes are off-centre, then the biaxial moment $I_{1^m2^n} = \int_F z_1^m z_2^n dF$ reaches its extremum when either

$$I_{1^{m-1}2^n} \text{ or } I_{1^m2^{n-1}} \text{ vanish}$$

3. UNIAXIAL MOMENTS OF POINT FORCES AND COUPLES

Tho product of a force and its arm is called statical moment of the force. Similar to area moments of geometric figures, the concept of a moment may be extended to other possible products of a force and its arm. The latter enters the formula not only in the first power, but also in any other power corresponding to the order of the moment.

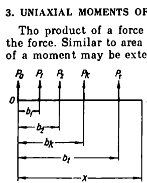

Fig. 3.1

The moment of the *n*-th order of a system of parallel forces $P_0, P_1, \ldots, P_k, \ldots, P_t$ (Fig. 3.1) at distances $b_1, b_2, \ldots, b_k, \ldots, b_t$ from the origin, respectively, is determined as a sum of products of each force and its arm, the power of the latter being equal to the order n of the moment

$$M_n = P_0x^n + P_1(x-b_1)^n + P_2(x-b_2)^n + \ldots + P_k(x-b_k)^n + \ldots + P_t(x-b_t)^n = \sum_{k=0}^{k=t} P_k(x-b_k)^n \tag{3.1}$$

Assuming $n = 0, 1, 2, 3, \ldots$, we obtain formulas for moments of the zero, first, second, etc. orders, respectively.

$$M_0 = P_0x^0 + P_1(x-b_1)^0 + P_2(x-b_2)^0 + \ldots + P_k(x-b_k)^0 + \ldots + P_t(x-b_t)^0 = \sum_{k=0}^{k=t} P_k$$

$$M_1 = \sum_{k=0}^{k=t} P_k(x-b_k)$$

$$M_2 = \sum_{k=0}^{k=i} P_k (x - b_k)^2$$

$$M_3 = \sum_{k=0}^{k=i} P_k (x - b_k)^3$$

These moments may be called: M_0—zero moment, M_1—first moment, M_2—second moment, etc.; the number of the moment corresponds to the power of the arm. The power of the arm in formula (3.1) is the same for all forces contributing to the moment.

The theory of moments of higher order does not set any limits for the order n of the moment; calculations may deal with, say, fifth moments M_5, seventh moments M_7, etc.

Let us now consider higher moments of point force couples. In order to determine the n-th moment of a force couple (Fig. 3.2*a*) we assume that the couple is formed by two forces P (Fig. 3.2*b*) applied at points separated by a distance Δ. The moment is equal to

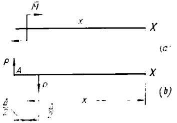

Fig. 3.2

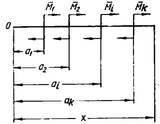

Fig. 3.3

$$\bar{M} = P\Delta$$

If the moment is kept constant, then $P \to \infty$ as $\Delta \to 0$. In this case from formula (3.1) we obtain

$$\begin{aligned} M_{n_{\Delta\to\infty}} &= P\left(x + \frac{\Delta}{2}\right)^n - P\left(x - \frac{\Delta}{2}\right)^n = \\ &= P_{\Delta\to 0}\left[x^n + nx^{n-1}\frac{\Delta}{2} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\left(\frac{\Delta}{2}\right)^2 + \dots\right] - \\ &- P_{\Delta\to 0}\left[x^n - nx^{n-1}\frac{\Delta}{2} + \frac{n(n-1)}{1\cdot 2}x^{n-2}\left(\frac{\Delta}{2}\right)^2 + \dots\right] = \\ &= P_{\Delta\to 0}nx^{n-1}\Delta + 2P\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^{n-3}\left(\frac{\Delta}{2}\right)^3 + \dots = \\ &= P\Delta\,[nx^{n-1} + \dots]_{\Delta\to 0} \end{aligned}$$

In the obtained expression, $P\Delta = \overline{M}$, and therefore

$$M_n = n\overline{M}x^{n-1} \tag{3.2}$$

because as $\Delta \to 0$ all subsequent terms in the formula vanish.

It can be observed now from comparison of (3.1) and (3.2) that the formula for the n-th moment of a force couple has the form of a derivative of the formula obtained for the moment of the same order of a point force.

Application of formulas (3.1) and (3.2) can be made much simpler if these formulas are presented in a different form by dividing both sides by a factorial product $n!$

This leads to the following formula for point forces

$$\frac{M_n}{n!} = \sum_{k=0}^{k=l} \frac{P_k (x-b_k)^n}{n!} \tag{3.3}$$

According to (3.2), for a system of point force couples $\overline{M}_1, \overline{M}_2, \ldots, \overline{M}_i, \ldots, \overline{M}_k$, applied at distances $a_1, a_2, \ldots, a_i, \ldots, a_k$, from the origin of coordinates (Fig. 3.3) the n-th moment about point X with abscissa x will be

$$\frac{M_n}{n!} = \sum_{i=1}^{i=k} \overline{M}_i \frac{(x-a_i)^{n-1}}{(n-1)!} \tag{3.4}$$

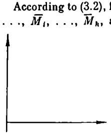

Fig. 3.4

Besides point forces and force couples, there are continuous distributed loads, which are represented in the schemes by loading areas of various forms. The formulas determining moments of higher order of all loads of this type with respect to various axes contain both the force factors (i.e., loads of a certain type) and arms of these force factors; this means that sign rules should be set for both factors.

According to generally accepted rules, we shall consider the vertical point load or area of a vertically directed distributed load positive when directed towards the positive end of the ordinate axis, i.e., upwards (Fig. 3.4), and negative in the opposite case. The arm of a force or loading area is positive when the moment axis is on their right.

If we follow this rule, then the positive arm of a force or loading area directed always from the force (or area) towards the moment axis, will coincide with the positive direction of the abscissa.

Thus, according to this rule the direction of a force and the direction of an arm are designated by signs, corresponding to the usual directions of coordinate axes.

For instance, the n-th moment of a vertical force P with an arm x with respect to the moment axis is

$$M_n = Px^n$$

When n is even, the sign of the arm x of the force does not matter and it may be given the same sign as that of the force P.

That is why an even moment of a force or loading area will be always positive, if the force or the loading area are directed along the positive half of the vertical axis (i.e., upwards), and negative in the opposite case regardless of the position of the moment point.

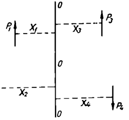

Fig. 3.5

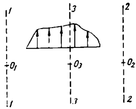

Fig. 3.6

As to the odd moments, their signs are determined according to the sign rules for bending moments, which, in the theory of strength of materials, correspond to the sign of the curvative of the elastic line produced by the moment.

According to Fig. 3.5, of the four presented forces, P_1 and P_3 should be included in the formulas with plus sigus, while P_2 and P_4 should be taken negative; the arms of these forces with respect to the moment axis have the following signs: x_1 and x_2 are positive, while x_3 and x_4 are negative. As a result, an odd moment of forces P_1 and P_4 about point O is positive, while the same moment of forces P_2 and P_3 is negative.

Regarding continuous loads (Fig. 3.6), it can be observed that any even moment of a positive load (directed upwards) is positive regardless of the position of the axis, because when determining an even moment the arm of a force or a loading area is raised to an even power, which results in a positive number; thus, the sign of an even moment is the same as the sign of the loading.

Moments of odd orders of a positive loading area will be negative with respect to an axis through point O_1 on line *1-1*, because all the elements of the loading area have negative abscissas.

Likewise, the odd moments with respect to an axis through point O_2 are always positive*.

As for the signs of moments of any odd order of the loading area or simply of the areas of geometrical figures with respect to axis *3-3* (Fig. 3.6) which cuts the area into two parts, it is impossible to say anything definite until the moment is calculated.

4. UNIAXIAL MOMENTS OF BALANCED AND UNBALANCED SYSTEMS

Consider the relationship between the moments of the same order relative to parallel axes.

The following notations are introduced: dF—element of area F (Fig. 4.1) of a plane geometric figure; x_O, x_A and x_B—distances from the area dF to axes *O-O*, *A-A* and *B-B*, respectively; c_A and c_B—distances between the central axis *O-O* and axes *A-A* and *B-B*; $c = c_B - c_A$—distance between axes *B-B* and *A-A*.

We shall now derive the expression for uniaxial moments of the n-th order of the area with respect to axes *A-A* and *B-B*. Let M_{A^n} and M_{B_n} be the moments of the n-th order relative to *A-A* and *B-B*, subscript n indicating the order of the moment and A and B indicating the axes involved. The moments of the n-th order with respect to axes *A-A* and *B-B* are determined by the following formulas

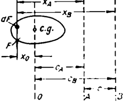

Fig. 4.1

$$M_{A^n} = \int_F x_A^n \, dF; \quad M_{B^n} = \int_F x_B^n \, dF$$

Hence,

$$\frac{M_{B^n}}{n!} = \frac{1}{n!} \int_F x_B^n \, dF = \frac{1}{n!} \int_F (x_A + c)^n \, dF =$$

$$= \frac{1}{n!} \int_F \Big[x_A^n + ncx_A^{n-1} + \frac{n(n-1)}{1 \cdot 2} c^2 x_A^{n-2} + \ldots + nc^{n-1} x_A +$$

$$+ c^n \Big] dF = \frac{1}{n!} \int_F x_A^n \, dF + c \frac{1}{(n-1)!} \int_F x_A^{n-1} \, dF +$$

$$+ \frac{c^2}{2} \cdot \frac{1}{(n-2)!} \int_F x_A^{n-2} \, dF + \ldots + \frac{c^n}{n!} \int_F dF$$

---

* Determination of moments of a load which is parallel to a certain axis becomes similar to determination of moments of plane geometric figures, if the new moment axis is introduced in the plane of the load instead of the usual axis which is perpendicular to the plane of the force.

and finally

$$\frac{M_{B^n}}{n!}=\frac{M_{A^n}}{n!}+c\,\frac{M_{A^{n-1}}}{(n-1)!}+\frac{c^2}{2!}\cdot\frac{M_{A^{n-2}}}{(n-2)!}+\frac{c^3}{3!}\cdot\frac{M_{A^{n-3}}}{(n-3)!}+\dots$$
$$\dots+\frac{c^{n-3}}{(n-3)!}\cdot\frac{M_{A^3}}{3!}+\frac{c^{n-2}}{(n-2)!}\cdot\frac{M_{A^2}}{2!}+\frac{c^{n-1}}{(n-1)!}\cdot\frac{M_{A^1}}{1!}+\frac{c^n}{n!}M_{A^0} \qquad (4.1)$$

Regardless of the position of the axis, the moment of the zero order of a distributed load is equal to the resultant of this load, i.e., it is represented by the loading area

$$M_{A^0}=M_{B^0}=F \qquad (4.2)$$

and for a system of point loads it is represented by the absolute value of the resultant

$$M_{A^0}=M_{B^0}=\sum P \qquad (4.3)$$

Formulas for relationships between moments with respect to two mutually parallel off-centre axes may be obtained from (4.1) as its particular cases. Taking $n=0, 1, 2, 3, 4, \dots$, the following formulas are obtained from (4.1) for moments of various orders.

Moment of zero order

$$M_{B^0}=M_{A^0}=\text{const}$$

(for a loading area $M_{B^0}=M_{A^0}=F$).

Similarly,

$$M_{B^1}=M_{A^1}+cM_{A^0} \qquad (4.4)$$

$$\frac{M_{B^2}}{2!}=\frac{M_{A^2}}{2!}+cM_{A^1}+\frac{c^2}{2!}M_{A^0} \qquad (4.5)$$

$$\frac{M_{B^3}}{3!}=\frac{M_{A^3}}{3!}+c\,\frac{M_{A^2}}{2!}+\frac{c^2}{2!}M_{A^1}+\frac{c^3}{3!}M_{A^0} \qquad (4.6)$$

$$\frac{M_{B^4}}{4!}=\frac{M_{A^4}}{4!}+c\,\frac{M_{A^3}}{3!}+\frac{c^2}{2!}\cdot\frac{M_{A^2}}{2!}+\frac{c^3}{3!}M_{A^1}+\frac{c^4}{4!}\cdot M_{A^0} \qquad (4.7)$$

where M_{B^1} is the moment of the first order, M_{B^2} —the moment of the second order, etc.

When axis A-A coincides with the central axis O-O, particular cases of formulas (4.1), (4.4)-(4.7) occur, representing the relationships between the moments with respect to two parallel axes, one of which is the central axis and the other is at a distance c from it. In these formulas the statical moment M_{O^1} with respect to the central axis is zero. Using M_x to denote the moments with respect to an arbitrary axis x, and M_0 to denote the moments with respect to the axis O, we come to the following formulas for the above mentioned relationships

$$M_{x^1} = cM_{O^0} \tag{4.8}$$

$$\frac{M_{x^2}}{2!} = \frac{M_{O^2}}{2!} + \frac{c^2}{2!} M_{O^0} \tag{4.9}$$

or

$$M_{x^2} = M_{O^2} + c^2 M_{O^0} \tag{4.9'}$$

$$\frac{M_{x^3}}{3!} = \frac{M_{O^3}}{3!} + c\frac{M_{O^2}}{2!} + \frac{c^3}{3!} M_{O^0} \tag{4.10}$$

or

$$M_{x^3} = M_{O^3} + 3cM_{O^2} + c^3 \tag{4.10'}$$

$$\frac{M_{x^4}}{4!} = \frac{M_{O^4}}{4!} + c\frac{M_{O^3}}{3!} + \frac{c^2}{2!} \cdot \frac{M_{O^2}}{2!} + \frac{c^4}{4!} M_{O^0} \tag{4.11}$$

or

$$M_{x^4} = M_{O^4} + 4cM_{O^3} + 6c^2 M_{O^2} + c^4 M_{O^0} \tag{4.11'}$$

where M_{x^1} is the moment of the first order, M_{x^2}—moment of the second order, etc.

The following formula refers to moments of arbitrary order

$$\frac{M_{x^n}}{n!} = \frac{M_{O^n}}{n!} + c\frac{M_{O^{n-1}}}{(n-1)!} + \frac{c^2}{2!} \cdot \frac{M_{O^{n-2}}}{(n-2)} + \frac{c^3}{3!} \cdot \frac{M_{O^{n-3}}}{(n-3)!} + \ldots +$$

$$+ \frac{c^{n-3}}{(n-3)!} \cdot \frac{M_{O^3}}{3!} + \frac{c^{n-2}}{(n-2)!} \cdot \frac{M_{O^2}}{2!} + \frac{c^n}{n!} \cdot M_{O^0} \tag{4.12}$$

If a self-balancing composite load consists of point forces, couples and continuous distributed loads, and the static equilibrium equations lead to $M_0 = 0$ and $M_1 = 0$ (with respect to any axis), then formulas (4.1) and (4.5)-(4.7) are simplified.

We shall now derive the formulas for the relationship between the moments of a self-balanced loading with respect to two arbitrary parallel axes.

Moments of zero, first, second, third and fourth orders may be determined by the following expressions

$$M_{B^0} = M_{A^0} = 0 \tag{4.13}$$

$$M_{B^1} = M_{A^1} = 0 \tag{4.14}$$

$$\frac{M_{B^2}}{2} = \frac{M_{A^2}}{2} = \text{const} \tag{4.15}$$

$$\frac{M_{B^3}}{3!} = \frac{M_{A^3}}{3!} + c\frac{M_{A^2}}{2} \tag{4.16}$$

$$\frac{M_{B^4}}{4!} = \frac{M_{A^4}}{4!} + c\frac{M_{A^3}}{3!} + \frac{c_2}{2!} \cdot \frac{M_{A^2}}{2} \tag{4.17}$$

etc.

The general expression for the moment of the n-th order is given by

$$\frac{M_{B^n}}{n!}=\frac{M_{A^n}}{n!}+c\,\frac{M_{A^{n-1}}}{(n-1)!}+\frac{c^2}{2!}\cdot\frac{M_{A^{n-2}}}{(n-2)!}+$$
$$+\frac{c^3}{3!}\cdot\frac{M_{A^{n-3}}}{(n-3)!}+\ldots+\frac{c^{n-3}}{(n-3)!}\cdot\frac{M_{A^3}}{3!}+\frac{c^{n-2}}{(n-2)!}\cdot\frac{M_{A^2}}{2!}\qquad(4.18)$$

If one of the parallel axes is the central axis, the formulas remain the same except that the subscript A should be replaced by O, and c should denote the distance between the arbitrary and central axes. Then

$$M_{x^0}=0\qquad(4.13')$$

$$M_{x^1}=0\qquad(4.14')$$

$$\frac{M_{O^2}}{2}=\text{const},\quad \frac{M_{x^2}}{2}=\text{const}\qquad(4.15')$$

$$\frac{M_{x^3}}{3!}=\frac{M_{O^3}}{3!}+c\,\frac{M_{O^2}}{2!}\qquad(4.16')$$

$$\frac{M_{x^4}}{4!}=\frac{M_{O^4}}{4!}+c\,\frac{M_{O^3}}{3!}+\frac{c^2}{2!}\cdot\frac{M_{O^2}}{2!}\qquad(4.17')$$

etc.

The general expression is

$$\frac{M_{x^n}}{n!}=\frac{M_{O^n}}{n!}+c\,\frac{M_{O^{n-1}}}{(n-1)!}+\frac{c^2}{2!}\cdot\frac{M_{O^{n-2}}}{(n-2)!}+$$
$$+\frac{c^3}{3!}\cdot\frac{M_{O^{n-3}}}{(n-3)!}+\ldots+\frac{c^{n-3}}{(n-3)!}\cdot\frac{M_{O^3}}{3!}+\frac{c^{n-2}}{(n-2)!}\cdot\frac{M_{O^2}}{2!}\qquad(4.18')$$

From formulas (4.13)-(4.18) the following four theorems concerning a plain system of balanced transverse forces can be deduced.

Theorem I. *The moment of the second order of all the forces (divided by 2!), with respect to any point of the plane is a constant value.*

Theorem II. *The moment of the third order (divided by 3!) is a linear function of the abscissa of the moment point.*

Theorem III. *The moment of the fourth order (divided by 4!) is a second order parabolic function of the abscissa of the moment point.*

Theorem IV. *The moment of the n-th order (divided by n!) is a parabolic function of the order (n — 2) of the abscissa of the moment point.*

If these theorems are supplemented by statical equilibrium equations (that the force resultant is equal to zero, i.e., $M_0=0$, and the statical moment resultant is equal to zero, $M_1=0$), then the following set of formulas will represent all these equations for a balanced system of forces

$$M_0 = 0, \quad M_1 = 0, \quad \frac{M_2}{2!} = C_1 = \text{const}$$

$$\frac{M_3}{3!} = C_1 x + C_2, \quad \frac{M_4}{4!} = C_1 \frac{x^2}{2} + C_2 x + C_3, \text{ etc.}$$

and, in the general form

$$\frac{M_n}{n!} = C_1 \frac{x^{n-2}}{(n-2)!} + C_2 \frac{x^{n-3}}{(n-3)!} + C_3 \frac{x^{n-4}}{(n-4)!} + \ldots + \\ + C_{n-3} \frac{x^2}{2} + C_{n-2} x + C_{n-1} \qquad (4.19)$$

where $C_1, C_2, C_3, \ldots, C_{n-2}, C_{n-1}$ are constant values depending on the magnitude of the forces as well as on their arrangement in the balanced system, x is the distance of an arbitrary axis from the central axis.

We shall consider the moments of higher order with respect to differently arranged parallel axes. Returning to Fig. 4.1, we shall determine the relationship between the moment of an arbitrary order of any geometrical figure and the position of the moment axis, when the latter is transferred parallel to its initial position. As previously, the moment of the n-th order with respect to axis B-B is determined by the following formula

$$\frac{M_{B^n}}{n!} = \frac{1}{n!} \int\limits_F x_B^n \, dF = \frac{1}{n!} (x_A + c)^n \, dF$$

Depending on the position of the moment axis, i.e.,on distance c, a moment of the n-th order reaches its extremum at such a value of c, when the derivative of the moment with respect to c equals zero, i.e.

$$\frac{\partial}{\partial c} \left(\frac{M_{B^n}}{n!} \right) = \frac{1}{n!} \int\limits_F \frac{\partial}{\partial c} (x_A + c)^n \, dF = \\ = \frac{1}{(n-1)!} \int\limits_F (x_A + c)^{n-1} dF = \frac{M_{B^{n-1}}}{(n-1)!} = 0 \qquad (4.20)$$

Similar calculations lead to

$$\frac{\partial}{\partial c^2} \left(\frac{M_{B^n}}{n!} \right) = \frac{M_{B^{n-2}}}{(n-2)!} \qquad (4.21)$$

After comparing these formulas with those derived previously from analysing the signs of the moments of even and odd orders, we come to the following conclusions.

All even moments of a positive loading area are positive, while the signs of the odd moments depend on the position of the moment axis. If the latter is situated so that the entire figure is on its left-hand side, then the odd moment with respect to this axis will be positive. And vice versa, if the figure is on the right-hand side of the axis, then the odd moment will be negative.

Therefore, any odd moment has an axis, with respect to which it becomes equal to zero. This axis is called *the central axis of the corresponding odd order*. For example, the central axis of the first order (the general central axis passing through the centre of gravity), the central axis of the third order, etc.

It may be concluded from formulas (4.18) and (4.19), that a positive moment M_{2n} of an even order $2n$ reaches its extremum with respect to the central axis of the order $(2n - 1)$. The second derivative M_{2n-2} has the same sign as the investigated function M_{2n}, i.e., $M_{2n-2} > 0$, then the extremum corresponds to the minimum value.

The statements proved above may be formulated as the following theorems*.

Theorem V. *A moment of any odd order* $(2n - 1)$ *has only one position of the moment axis, with respect to which its value* M_{2n-1} *becomes equal to zero. Such an axis is called the central axis of the order* $(2n - 1)$.

Theorem VI. *A moment* M_{2n} *of any even order* $(2n)$ *of any area of a geometric figure reaches its minimum absolute value with respect to the central axis of the order* $(2n - 1)$. *In particular, the moment of inertia of an area of a geometric figure is the smallest with respect to the axis, which passes through the centre of gravity.*

Theorem VII. *If a given composite load system (forces, force couples, distributed loads) is symmetrical, then any odd moment with respect to the axis of symmetry is equal to zero. A moment of any even order reaches its minimum absolute value with respect to the axis of symmetry.*

The concept of moments of higher order in three-dimensional space may be extended to various vectorial quantities given in space.

5. UNIAXIAL MOMENTS OF AREAS OF SIMPLE FIGURES

Relationship Between Uniaxial Moments Relative to Parallel Axes

Consider an area F (Fig. 5.1) of arbitrary shape and two parallel axes in the plane of the figure, a central axis *O-O* and axis *1-1* set at a distance c from the central axis. The distances from an area element dF to the axes *O-O* and *1-1* are denoted by x_0 and x_1, respectively. The moment of the n-th order of area F with respect

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[*] Similar considerations are true not only for areas of plane figures, but also for composite loads consisting of forces, force couples and distributed loads, and in general for combination of any force parameters.

to axis *1-1*, divided by $n!$, is determined by the following formula*

$$\frac{M_{1^n}}{n!} = \frac{1}{n!}\int_F x_1^n \, dF$$

But $x_1 = x_0 + c$, therefore

$$\frac{M_{1^n}}{n!} = \int_F \frac{(x_O + c)^n}{n!} dF = \int_F \Big[\frac{x_O^n}{n!} + c\frac{x_O^{n-1}}{(n-1)!} + \\ + \frac{c^2}{2!}\cdot\frac{x_O^{n-2}}{(n-2)!} + \frac{c^3}{3!}\cdot\frac{x_O^{n-3}}{(n-3)!} + \dots + \frac{c^{n-3}}{(n-3)!}\cdot\frac{x_0^3}{3!} + \\ + \frac{c^{n-2}}{(n-2)!}\cdot\frac{x_0^2}{2!} + \frac{c^{n-1}}{(n-1)!} x_O + \frac{c^n}{n!}\Big] dF$$

By introducing moments relative to the central axis and remembering that the static moment relative to the central axis is equal to zero we obtain a formula expressing the relation between moments of the n-th order relative to two parallel axes, one of which is the central axis

$$\frac{M_{1^n}}{n!} = \frac{M_{O^n}}{n!} + c\frac{M_{O^{n-1}}}{(n-1)!} + \frac{c^2}{2!}\cdot\frac{M_{O^{n-2}}}{(n-2)!} + \\ + \frac{c^3}{3!}\cdot\frac{M_{O^{n-3}}}{(n-3)!} + \dots + \frac{c^{n-3}}{(n-3)!}\cdot\frac{M_{O^3}}{3!} + \\ + \frac{c^{n-2}}{(n-2)!}\cdot\frac{M_{O^2}}{2!} + \frac{c^n}{n!} F \tag{5.1}$$

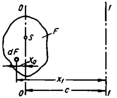

Fig. 5.1

Formulas for the moments of the second, third, fourth, etc. order can be derived from (5.1) by substituting $n = 2, 3, 4, \dots$.

From here for brevity, moments of the first, second, third, etc. orders will be called the first, second, third, etc. moments, respectively.

For the second moments (when $n = 2$)

$$\frac{M_{1^2}}{2!} = \frac{M_{O^2}}{2!} + \frac{c^2}{2!} F \tag{5.2}$$

This expression is well-known as the formula determining the relation between the moments of inertia with respect to two parallel axes, one of which is the central axis.

For the third moments (when $n = 3$)

$$\frac{M_{1^3}}{3!} = \frac{M_{O^3}}{3!} + c\frac{M_{O^2}}{2!} + \frac{c^3}{3!} F \tag{5.3}$$

---

* The formula expressing the relationship between the moments is derived here independently of formulas of the previous section, although it is a particular case of those formulas.

or

$$M_{1^3} = M_{0^3} + 3cM_{0^2} + c^3F \tag{5.4}$$

For the fourth moments (when $n = 4$)

$$\frac{M_{1^4}}{4!} = \frac{M_{0^4}}{4!} + c\frac{M_{0^3}}{3!} + \frac{c^2}{2!}\cdot\frac{M_{0^2}}{2!} + \frac{c^4}{4!}F \tag{5.5}$$

or

$$M_{1^4} = M_{0^4} + 4cM_{0^3} + 6c^2M_{0^2} + c^4F\text{, etc.} \tag{5.6}$$

Determination of Uniaxial Moments of Rectangular and Triangular Areas

We shall now calculate moments of higher order of areas of very simple figures, which may represent cross-sectional areas, loading areas, various diagrams and distributed loads. Distributed loads are usually represented in engineering by loading areas of rectangular, triangular, and trapezoidal form; the latter may be always divided into rectangles and triangles. More complicated loading areas described by parabolic segments or figures with trigonometric curves are seldom used in engineering. However, even such complicated figures do not lead to any difficulties in determining moments of higher order.

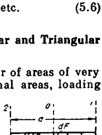

Fig. 5.2

For example, let us determine higher moments of the positive area of a rectangle (Fig. 5.2) with respect to parallel axes *1-1* and *2-2*. In order to determine the *n*-th moment of area F an infinitely small section dF of the rectangle is taken and the following formula is derived

$$\frac{M_{1^n}}{n!} = \int\limits_{u=0}^{u=a} \frac{u^n}{n!}\,dF = \int\limits_{u=0}^{u=a} q\,\frac{u^n}{u!}\,du = \frac{qa^{n+1}}{(n+1)!} \tag{5.7}$$

Taking $n = 0, 1, 2, 3, \ldots$, we find the area, statical moment, moment of inertia, moment of the third order, etc., about axis *1-1*

$$\left.\begin{aligned} &M_0 = F = qa, \quad M_1 = \frac{qa^2}{2!} \\ &\frac{M_2}{2!} = \frac{qa^3}{3!}, \quad \frac{M_3}{3!} = \frac{qa^4}{4!}\text{, etc.} \end{aligned}\right\} \tag{5.8}$$

The moments in these formulas are always divided by corresponding factorials (it is important to remember that $0! = 1$)*.

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* Since $n! = \frac{(n+1)!}{(n+1)}$, when $n=0$, we arrive at $0! = \frac{1!}{1} = 1$. This generalization of the concept of a factorial is widely used in the theory of special functions.

There is no need to calculate moments with respect to axis *2-2*, because due to the symmetry of the area the moments with respect to axes *1-1* and *2-2* have the same absolute values. All odd moments with respect to axis *2-2* are negative, i.e., $M_{2^{2n+1}} = -M_{1^{2n+1}}$ for any n, all even moments are positive in this case, i.e., $M_{2^{2n}} = +M_{1^{2n}}$ because they are of the same sign as the area.

All odd moments with respect to the central axis *O-O* (which is an axis of symmetry of the given area) are zero, while even moments

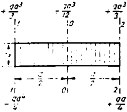

Fig. 5.3

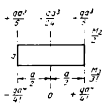

Fig. 5.4

are positive. To calculate them, we divide the rectangle in two even rectangles of length $a/2$. According to (5.7), we obtain

$$\frac{M_{0^n}}{n!} = 2\,\frac{q\left(\frac{a}{2}\right)^{n+1}}{(n+1)!} = \frac{qa^{n+1}}{(n+1)!\,2^n} \tag{5.9}$$

This formula leads to the following formulas of the area, moment of inertia (divided by 2!), moment of the fourth order (divided by 4!), etc., when $n = 0, 2, 4 \ldots$

$$\left.\begin{aligned} &M_{0^0} = F = qa \\ &\frac{M_{0^2}}{2!} = \frac{qa^3}{3!\cdot 2^2} = \frac{qa^3}{24} \\ &\frac{M_{0^4}}{4!} = \frac{qa^5}{5!\cdot 2^4}\text{, etc.} \end{aligned}\right\} \tag{5.10}$$

The values of the second and third moments are shown in Fig. 5.3, relative to the two most significant axes lying in the same plane; the second moments are written above the rectangle and the third moments, below. Figure 5.4 shows the same moments divided by 2! and 3!, respectively.

Consider moments of higher order with respect to the most significant axes for a triangle with its apex pointing to the right and an angle coefficient $\tan\alpha = k$ (Fig. 5.5). Taking an infinitely small area of the triangle $dF = p\,du$, where $p = ku$, we obtain the following formula for the n-th moment of the area with respect to axis *1-1*

$$M_{1^n} = \int\limits_F u^n\,dF = k \int\limits_{u=0}^{u=a} u^{n+1}\,du = \frac{ka^{n+2}}{(n+2)} \quad (5.11)$$

or

$$\frac{M_{1^n}}{n!} = \frac{k(n+1)\,a^{n+2}}{(n+2)!} \qquad (5.12)$$

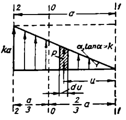

Fig. 5.5

To calculate the n-th moment (divided by n!) with respect to axis *2-2*, we consider the area of the triangle as the difference of the area of a rectangle of height $q = ka$ and that of a supplementary triangle, the outline of which is shown in Fig. 5.5 by dashed lines. Using formulas (5.7) and (5.12), we obtain

$$\frac{M_{2^n}}{n!} = (-1)^n \left[\frac{kaa^{n+1}}{(n+1)!} - \frac{k(n+1)\,a^{n+2}}{(n+2)!}\right] = (-1)^n \frac{ka^{n+2}}{(n+2)!} \quad (5.13)$$

According to this formula, all even moments are positive and odd moments are negative.

The value of the moment (without the factorial denominator) is given by

$$M_{2^n} = (-1)^n \frac{ka^{n+2}}{(n+1)(n+2)} \qquad (5.14)$$

This formula may be used to calculate the area, statical moment, second moment (moment of inertia), third moment, fourth moment, etc., corresponding to $n = 0, 1, 2, 3, 4, \ldots$, respectively

$$F = +\frac{ka^2}{2}, \quad S_2 = M_{2^1} = -\frac{ka^3}{2\cdot 3} = -\frac{ka^3}{6},$$

$$M_{2^2} = +\frac{ka^4}{3\cdot 4} = \frac{ka^4}{12}, \quad M_{2^3} = -\frac{ka^5}{4\cdot 5} = -\frac{ka^5}{20},$$

$$M_{2^4} = +\frac{ka^6}{5\cdot 6} = \frac{ka^6}{30} \ldots \qquad (5.15)$$

Formula (5.4) may be used to determine the third moment with respect to the central axis *O-O*, taking *2-2* as the initial axis and $c = -\frac{a}{3}$

$$M_{O^3} = M_{2^3} - 3cM_{O^2} - c^3F = -\frac{ka^5}{20} + 3\,\frac{a}{3}\cdot\frac{ka^4}{36} + \left(\frac{a}{3}\right)^5\cdot\frac{ka^5}{3} = -\frac{ka^5}{270} \qquad (5.16)$$

The central third moment could be determined by direct integration (as we would obtain the general formula for the n-th moment), but in view of the relative amount of calculations the method of deriving formula (5.16) was found most appropriate; moreover, the only central moments necessary will be those of the second and third orders.

Calculations carried out from the derived formulas result in values presented in Figs. 5.6 and 5.7; the second moments are written on top of the triangles, the third moments—under them. Figure 5.6 presents the moments, while Fig. 5.7 gives the same moments divided by corresponding factorials.

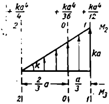

Fig. 5.6

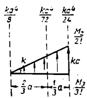

Fig. 5.7

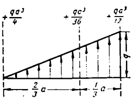

Fig. 5.8

If the largest ordinate $q = ka$ is given instead of the angle coefficient k, then the values of the second and third moments with respect to the three most significant axes will be as shown in Fig. 5.8.

It can be concluded from Figs. 5.6-5.8 that according to the sign rule, the second even moments of a positive area of a triangle are positive with respect to all the axes. As for the sign of the third (odd) moments, this depends on the sign of the load and the position of the moment axis.

The moment of the third order of a positive area of a triangle will always be positive with respect to the axis passing through the vertical base of the triangle (axis *1-1* in Figs. 5.5 and 5.6). This follows from the direction of rotation about the moment axis and from the fact that all the surface elements of a triangle have positive arms relative to this axis. Similar considerations lead to the conclusion, that moments of the third order with respect to a vertical axis passing through the apex of a positive triangle are negative (the extreme left axes in Figs. 5.5-5.8).

To determine the sign of the third central moment of a triangle is more difficult. The statical moment with respect to the central axis is zero, but when finding the third moment, all the moment arms of the surface elements are raised not to the first power, but to the third; this is why in the integrand of the expression the products of the elements arranged on the left side will predominate since they are, comparatively speaking, farther from the central axis than the elements on the right. As a result, the central axis will be in the same condition relative to the sign of the moment as

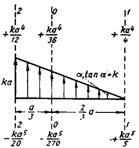

Fig. 5.9

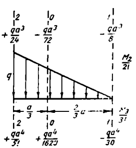

Fig. 5.10

the axis passing through its base: the sign of the third moment (in general, any odd moment except the first) of the area of the triangle with respect to the central axis always coincides with the sign of the third moment with respect to the axis passing through the vertical base of the triangle, and opposite to the sign of the moment of the same order with respect to the vertical axis passing through the apex of the triangle.

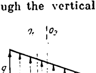

Fig. 5.11

The above mentioned considerations are illustrated by Fig. 5.9, where moments M_2 are written above the triangle and moments M_3—under it for a positive area of a triangle with its apex pointing to the right. Fig. 5.10 has moments, divided by the corresponding factorials $\left(\frac{M_2}{2!}\right.$ above the triangle and $\left.\frac{M_3}{3!}\text{ under it}\right)$.

In conclusion let us find the position of the central axis of the third order, i.e., such an axis, with respect to which the third moment of the area of a triangle is zero. Let a triangle (Fig. 5.11) have base a and height q and O_1-O_1 be the central axis i.e., the central axis of the first order, O_3-O_3 —the central (zero) axis of the third order. Then, according to formula (5.4), the third moment with respect to the zero axis of the third order is determined by the following expression

$$M_{O_3^3} = M_{O_1^3} + 3cM_{O_1^2} + c^3F$$

where c is the distance between zero axes of the first and third order. After substitutions, the following expression is derived

$$M_{O_3} = -\frac{qa^4}{270} + 3c\left(\frac{qa^3}{36}\right) + c^3F = 0$$

The only real root of this cubic equation is given by the following expression

$$c = \frac{a}{3}\sqrt[3]{\frac{1}{10} + \frac{3\sqrt{6}}{20}} + \frac{a}{3}\sqrt{\frac{1}{10} - \frac{3\sqrt{6}}{20}} \tag{5.17}$$

or

$$c \approx \frac{a}{3}\sqrt[3]{0.4674235} + \frac{a}{3}\sqrt[3]{-0.267435} = 0.0439a = \frac{a}{22.78} \tag{5.18}$$

The distance between the zero axis of the third order and the vertical cathetus of the triangle is

$$c_3 = c + \frac{a}{3} = \frac{1}{2.6511}a \tag{5.19}$$

Similar procedures help in establishing the position of the zero axis of the fifth order, which is shifted farther to the right of the figure than the O_3-O_3 axis, etc.

6. UNIAXIAL MOMENTS OF AREAS OF MORE COMPLICATED FIGURES

In Section 5 expressions were derived for the second and third moments of uniform and non-uniform (uniformly increasing or decreasing) loads most commonly used in engineering practice.

The present section deals with compound loads, the loading areas of which are described by trigonometrical and parabolic curves.

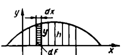

Fig. 6.1

Complicated loads are considered in dynamics of machine components and structural elements and in solving other problems concerning the behaviour of elastic media. Loading areas with trigonometric and parabolic contours are the most general types. This follows from the fact that the loads studied in Sec. 5, as well as certain other loads, may be considered particular cases of a loading area outlined by a higher order parabola (see examples in Sec. 7).

Let us determine the moments of higher order of the area of a sinusoid. Consider the n-th moment with respect to axis y-y of a part of the area between the sinusoid and the abscissa (Fig. 6.1). Let the sinusoid equation be

$$y = h\sin\left(\alpha + \frac{\pi x}{l}\right)$$

Then the required n-th order moment divided by $n!$ is given by the following formula

$$\frac{M_n}{n!}=\int\limits_F \frac{x^n}{n!}\,dF$$

where dF is the area of the shaded element equal to

$$dF=y\,dx=h\sin\left(\alpha+\frac{\pi x}{l}\right)dx$$

After substituting and integrating by parts we obtain

$$\frac{M_n}{n!}=\int\limits_{x=0}^{x=a}\frac{x^n}{n!}h\sin\left(\alpha+\frac{\pi x}{l}\right)dx=h\int\limits_{x=0}^{x=a}\sin\left(\alpha+\frac{\pi x}{l}\right)d\,\frac{x^{n+1}}{(n+1)!}$$

hence

$$\frac{M_n}{n!}+\frac{\pi^2}{l^2}\,\frac{M_{n+2}}{(n+2)!}=h\sin\left(\alpha+\frac{\pi a}{l}\right)\frac{a^{n+1}}{(n+1)!}-$$
$$-\frac{\pi h}{l}\cos\left(\alpha+\frac{\pi a}{l}\right)\frac{a^{n+2}}{(n+2)!} \qquad (6.1)$$

Formula (6.1) is a recurrent one allowing to determine the moment of order $(n+2)$ in terms of the n-th moment.

In order to determine the area of the shaded element and its statical moment relative to the axis, the following integrals are to be calculated

$$M_O=F=\int\limits_{x=0}^{x=a}y\,dx=h\int\limits_{x=0}^{x=a}\sin\left(\alpha+\frac{\pi x}{l}\right)dx=$$
$$=-\int\limits_{x=0}^{x=a}\frac{lh}{\pi}\cos\left(\alpha+\frac{\pi x}{l}\right)=\frac{lh}{\pi}\left[\cos\alpha-\cos\left(\alpha+\frac{\pi a}{l}\right)\right] \qquad (6.2)$$

and

$$M_1=S=\int\limits_{x=0}^{x=a}yx\,dx=h\int\limits_{x=0}^{x=a}\sin\left(\alpha+\frac{\pi x}{l}\right)x\,dx=$$
$$=-\frac{lh}{\pi}a\cos\left(\alpha+\frac{\pi a}{l}\right)+\frac{l^2h}{\pi^2}\sin\left(\alpha+\frac{\pi a}{l}\right)-\frac{l^2h}{\pi^2}\sin\alpha \qquad (6.3)$$

By employing formula (6.1) moments of any order may be determined. All even moments are consequently determined by (6.1), if, according to (6.2), a zero moment is taken for the initial one. Similarly, all odd moments are determined, if, according to (6.3), the initial moment is a moment of the first order.

Formulas (6.1)-(6.3) are simplified if a half-wave sinusoid is considered by taking $\alpha = 0$ and $a = l$

$$\frac{M_n}{n!}+\frac{\pi^2}{l^2}\cdot\frac{M_{n+2}}{(n+2)!}=\pi h\,\frac{l^{n+1}}{(n+2)!} \tag{6.4}$$

$$M_0=F=2\,\frac{lh}{\pi} \tag{6.5}$$

$$M_1=S=\frac{l^2h}{\pi} \tag{6.6}$$

Let ABC be a half-wave sinusoid symmetrical about the ordinate y (Fig. 6.2). In this case for the right half of the figure OBC, according to formula (6.5), the area is $\overline{F}=lh/\pi$. According to the equation of the curve

$$y=h\sin\left(\frac{\pi}{2}+\frac{\pi x}{l}\right)$$

and, when $\alpha=\pi/2$ and $a=l/2$, from (6.1) we obtain

$$\frac{M_n}{n!}+\frac{\pi^2}{l^2}\cdot\frac{M_{n+2}}{(n+2)!}=\pi\,\frac{h}{l}\cdot\frac{l^{n+2}}{2^{n+2}\,(n+2)!} \tag{6.7}$$

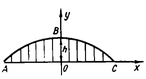

Fig. 6.2

Substituting $n=0$ into formula (6.7) we have

$$\overline{F}+\frac{\pi^2}{l^2}\cdot\frac{\overline{M}_2}{2}=\frac{\pi hl}{8}$$

from which

$$\frac{\overline{M}_2}{2}=hl^3\,\frac{\pi^2-8}{8\pi^3}$$

Since the area is symmetrical about the ordinate axis, the moment of inertia of area ABC relative to this axis is

$$\frac{M_2}{2}=2\,\frac{\overline{M}_2}{2}=hl^3\,\frac{\pi^2-8}{4\pi^3} \tag{6.8}$$

$$M_2=hl^3\,\frac{\pi^2-8}{2\pi^3} \tag{6.8'}$$

Due to the symmetry of the area, all odd moments relative to y are zero.

Thus, formula (6.5) determines the area of a sinusoid, formula (6.6)—the moment of inertia of this area, and the moments of higher order are calculated by formula (6.7).

Formulas (5.2) and (5.4) may be used to determine second and third moments of a sinusoid with respect to any vertical axis.

Let us consider higher moments of areas with parabolic contours.

The moment of the n-th order of a parabolic segment bounded by an m-th order parabolic curve $y=q\,\frac{u^m}{m!}$ (Fig. 6.3) with respect to axis *2-2* is expressed by the following formulas

$$\frac{M_n}{n!}=\frac{1}{n!}\int_F (a-u)^n\,dF=\frac{1}{n!}\int_{u=0}^{u=a}(a-u)^n\,q\frac{u^m}{m!}\,du=$$

$$=\frac{q}{m!\,n!}\int_{u=0}^{u=a}(a-u)^n u^m\cdot du \tag{a}$$

The integral is calculated by using the transformation formula for reducing an $(n+1)$-multiple integral to a prime integral

$$I=\int_0^a\int_0^a\dots\int_0^a f(u)\,du^{n+1}=$$

$$=\frac{1}{n!}\int_0^a f(u)(a-u)^n\,du \tag{b}$$

Assuming $u^m=f(u)$ in formula (a), we derive from formula (b)

$$\frac{M_n}{n!}=\frac{q}{m!\,n!}\int_0^a (a-u)^n f(u)\,du=$$

$$=\frac{q}{m!}\int_0^a du\int_0^a du\dots\int_0^a u^m\,du=$$

$$=\frac{qa^{m+n+1}}{(m+n+1)!} \tag{6.9}$$

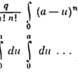

Fig. 6.3

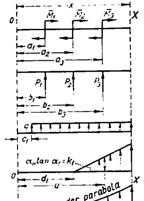

Fig. 6.4

When $m=0$ this formula coincides with formula (5.7) for the n-th moment of the area of a rectangle; when $m=1$, with formula (5.13) for the n-th moment of the triangle; when $m=2$, for the n-th moment of the area of a segment of a square parabola

$$\frac{M_n}{n!}=\frac{qa^{n+3}}{(n+3)!},\ \text{etc.} \tag{6.10}$$

The n-th moment of a compound load shown in Fig. 6.4 is given by the following:

$$\frac{M_n}{n!} = \sum \overline{M}_i \frac{(x-a_i)^{n-1}}{(n-1)!} + \sum P_i \frac{(x-b_i)^n}{n!} + \sum q_i \frac{(x-c_i)^{n+1}}{(n+1)!} + \\ + \sum k_i \frac{(x-d_i)^{n+2}}{(n+2)!} + \sum s_i \frac{(x-e_i)^{n+3}}{(n+3)!} + \sum t_i \frac{(x-g_i)^{n+4}}{(n+4)!} \qquad \text{(I)}$$

7. UNIAXIAL MOMENTS OF COMPOUND LOADS

Relationship Between Higher Moments and Multiple Integrals

Suppose a compound load acts upon a section OX of a beam (Fig. 6.4); some of the force parameters may be support reactions or reactive moments. It does not matter for the calculations, whether the forces or force couples are active or reactive.

In deriving the main formulas the force parameters are considered positive if the forces are directed upwards and the force couples turn the beam clockwise.

The load consists of point force couples $M_1, M_2, M_3, \ldots$, acting on sections with abscissas $a_1, a_2, a_3, \ldots$; of point forces $P_1, P_2, P_3, \ldots$, acting on sections with abscissas $b_1, b_2, b_3, \ldots$; of uniformly distributed loads $q_1, q_2, q_3, \ldots$, acting on sections with abscissas $c_1, c_2, c_3, \ldots$, and vanishing at the right-hand end of the beam; of triangular loading areas with $k_1 = \tan\alpha_1$, $k_2 = \tan\alpha_2$, $k_3 = \tan\alpha_3, \ldots$, beginning at sections with abscissas $d_1, d_2, d_3, \ldots$; of parabolic loading areas, bounded by second order parabolas

$$y = s_1 \frac{(u-e_1)^2}{2}, \quad y = s_2 \frac{(u-e_2)^2}{2}, \\ y = s_3 \frac{(u-e_3)^2}{2} \ldots$$

(where $s_1, s_2, s_3, \ldots$ are the second derivatives of the ordinates with respect to the abscissa, and $e_1, e_2, e_3, \ldots$ are abscissas of the second order parabola apexes); of loading areas bounded by third order parabolas

$$y = t_1 \frac{(u-g_1)^3}{3!}, \quad y = t_2 \frac{(u-g_2)^3}{3!}, \quad y = t_3 \frac{(u-g_3)^3}{3!} \ldots$$

(where $t_1, t_2, t_3, \ldots$ are the third derivatives of the ordinates with respect to the abscissa, and $g_1, g_2, g_3, \ldots$ are the abscissas of the cubic parabola apexes).

The bending moment M_X at a section X with an abscissa x is known to be a statical moment (i.e., a moment of the first order) of all the loads acting within section OX, while the transversal force Q_X at the same section is a resultant of all the forces within OX, or, in other words, is a moment of zero order of all the loads applied to the left of section X. Hence,

$$M_X = M_1, \quad Q_X = M_0 \tag{7.1}$$

However, since $Q_X = \frac{dM_X}{dx}$, it follows that $\frac{d}{dx} M_1 = M_0$.

The integral-differential relationship derived for the moments of the first and zero order is a particular case of the general relationship between moments of various orders, based on the relationship between moments of higher order and multiple integrals.

Indeed, let us examine a simple differential equation

$$\frac{d^{n+1}y}{dx^{n+1}} = f(x)$$

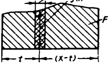

Fig. 7.1

By applying the formula of integrating a higher order equation in quadratures and by converting the multiple integral into a prime one according to the Cauchy formula, we obtain

$$y = \sum_{i=1}^{i=n} c_i \frac{x_i}{i!} + \frac{1}{n!} \int_{t=0}^{t=x} (x-t)^n f(t)\, dt \tag{7.2}$$

This formula means that all members of the right-hand side may be considered moments of higher order.

Indeed, if function $f(t)$ is plotted (Fig. 7.1), then the expression $f(t)\, dt$ under the integral sign represents an infinitely small surface element dF and the integral $\int_{t=0}^{t=x} (x-t)^n f(t)\, dt$ is the moment of the n-th order of the total area F, which lies between vertical lines $t = 0$ and $t = x$, with respect to the latter vertical line.

On this basis, the following theorem can be formulated.

Theorem VIII. *An $(n+1)$-multiple integral of any function $f(t)$ is equal to an n-multiple moment divided by n! of the area bounded by the $f(x)$ curve, the abscissa and the vertical lines.*

According to Theorem VIII, the following expressions are valid if the first, second, third, ... integrals of any function $f(t)$ are designated by $I_1, I_2, I_3, \ldots$

$$I_1 = \frac{M_0}{0!} = M_0 = F, \quad I_2 = \frac{M_1}{1!} = M_1, \quad I_3 = \frac{M_2}{2!}; \quad I_4 = \frac{M_3}{3!} \tag{7.3}$$

and, in general,

$$I_{n+1} = \frac{M_n}{n!} \tag{II}$$

According to (II), the order of the moment increases by a unit after integration and decreases by a unit after differentiation.

Indeed, if

$$I_{n+1}=\frac{M_n}{n!}$$

then

$$I_n=\frac{M_{n-1}}{(n-1)!}$$

However, $I_{n+1}=\int I_n\,dx$, hence

$$\frac{M_n}{n!}=\int\frac{M_{n-1}}{(n-1)!}\,dx \tag{7.4}$$

and, vice versa,

$$\frac{\partial}{\partial x}\left(\frac{M_n}{n!}\right)=\frac{M_{n-1}}{(n-1)!} \tag{7.5}$$

The quantities listed below may be determined by comparing formulas (7.1) and (7.3) written for section X with abscissa x (Fig. 6.4), by introducing symbolic moments of negative orders and by transition from the bending moment M_X to the transversal force Q_X and further to the load intensity q_X and its derivatives with respect to the abscissa, i.e.*,

$$\frac{dq_X}{dx}=k_X,\quad \frac{d^2q_X}{dx^2}=s_X,\quad \frac{d^3q_X}{dx^3}=t_X$$

The relationship between the computed quantities is then given by the following formulas

$$\left.\begin{aligned}
&M_X=M_1,\quad Q_X=\frac{dM_X}{dx}=\frac{dM_1}{dx}=M_0\\
&q_X=\frac{dQ_X}{dx}=\frac{d^2M_X}{dx^2}=M_{-1};\ k_X=\frac{dq_X}{dx}=\\
&\quad=\frac{d^2Q_X}{dx^2}=\frac{d^3M_X}{dx^3}=M_{-2}\\
&s_X=\frac{dk_X}{dx}=\frac{d^2q_X}{dx^2}=\frac{d^3Q_X}{dx^3}=\frac{d^4M_X}{dx^4}=M_{-3}\\
&t_X=\frac{ds_X}{dx}=\frac{d^2k_X}{dx^2}=\frac{d^3q_X}{dx^3}=\frac{d^4Q_X}{dx^4}=\frac{d^5M_X}{dx^5}=M_{-4}
\end{aligned}\right\} \tag{7.6}$$

---

* The subscript X means that the corresponding value is determined for section X (with abscissa x); small letters designate axes: for example, M_x, M_y and M_z are moments with respect to axes x, y and z, and Q_y, Q_z are projections of forces on axes y and z, etc.

According to formula (I), formulas (7.6) may be presented in the following form

$$M_X = M_1 = \sum M_i + \sum P_i (x - e_i) + \sum q_i \frac{(x - c_i)^2}{2} + \\ + \sum k_i \frac{(x - d_i)^3}{3!} + \sum s_i \frac{(x - c_i)^4}{4!} + \sum t_i \frac{(x - g_i)^5}{5!} \tag{7.7}$$

$$Q_X = M_0 = \sum P_i + \sum q_i (x - c_i) + \sum k_i \frac{(x - d_i)^2}{2} + \\ + \sum s_i \frac{(x - e_i)^3}{3!} + \sum t_i \frac{(x - g_i)^4}{4!} \tag{7.8}$$

$$q_X = M_{-1} = \sum q_i + \sum k_i (x - d_i) + \sum s_i \frac{(x - e_i)^2}{2} + \\ + \sum t_i \frac{(x - g_i)^3}{3!} \tag{7.9}$$

$$k_X = M_{-2} = \sum k_i + \sum s_i (x - e_i) + \sum t_i \frac{(x - g_i)^2}{2} \tag{7.10}$$

$$s_X = M_{-3} = \sum s_i + \sum t_i (x - g_i) \tag{7.11}$$

$$t_X = M_{-4} = \sum t_i \tag{7.12}$$

The general expression for the n-th moment (divided by $n!$) of a compound load with respect to the vertical X-X with abscissa x is given by formula (I) in Sec. 6.

Formula (I) is one of the basic formulas in the theory of moments of higher order and is widely used in the following sections of the book.

All the formulas of the present section allow for the influence of various statical factors forming a compound load which is presented in Fig. 6.4. It should be noted here, that the expression for the bending moment in (7.7) starts with concentrated moments; similarly, the first term on the right-hand side of formula (7.8) for the transversal force takes into account the point loads, etc. Passing from (7.7) to (7.8) and then to (7.9) and so on, we truncate the first term of the formula, and the number of terms in the formula gradually decreases. On the contrary, when gradually passing from formula (7.12) to formula (7.7), with increasing order of moments the number of terms increases every time by one.

Formulas for the second, third, fourth,... moments contain one, two, three,... terms more, than formula (7.7) for the first moment.

New terms appearing in the formulas during integration of formula (7.7) take into account many factors besides the usual forces shown in Fig. 6.4. That is why new terms in the series should be introduced so as to take into account these factors while expanding formula (I) for the moments of second or higher orders (see Sec. 8).

Moments of Higher Order for Various Loads

For various engineering problems which actually require solving differential equilibrium equations, the moment technique may be used in two ways: (1) a certain problem may be first solved by a non-momental method for a unit force loading (the method of influence lines) and then the results may be generalized and expanded to various loads by means of moment designations, for example, by using formula (I) of Sec. 6; (2) no particular loading at all may be involved in deriving the solution, the latter being reached by means of integration and, in some cases, differentiation procedures with moment designations (or, to be more precise, moment functions). In the second case the solution obtained in terms of higher moments may be transformed by expanding the moment terms for the given particular load. In order to make the task easier and to make use of the moment technique when evaluating statically indeterminate beams and frames, it is advisable to have formulas and reference tables ready for various loadings, so that there is no need to derive moment formulas for each load. The formulas presented in this article for various loads will be used throughout.

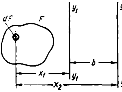

Fig. 7.2

It should be noted, that when using moment formulas it is advisable to calculate moments and divide them by factorials corresponding to the order of the moment. A moment of the n-th order of an area, divided by $n!$, is expressed by the following formula

$$\frac{M_n}{n!}=\frac{1}{n!}\int_F x^n dF \tag{7.13}$$

Let us determine the relationship between the moments with respect to mutually parallel axes. Let axes y_1-y_1 and y_2-y_2 be situated at distance b one from the other (Fig. 7.2); variable distances between an area element dF and the axes y_1-y_1 and y_2-y_2 are, correspondingly, x_1 and $x_2 = x_1 + b$.

Using these notations, a moment of the n-th order of area F of the total figure with respect to axis y_1-y_1, divided by $n!$, is given by the following formula

$$\frac{M_{1^n}}{n!}=\int_F \frac{x_1^n}{n!} dF \tag{7.14}$$

The moment of the n-th order of area F with respect to axis y_2-y_2, divided by $n!$, is determined by the following formula

$$\frac{M_{2^n}}{n!} = \int_F \frac{x_2^n}{n!}\,dF = \int_F \frac{(x_1+b)^n}{n!}\,dF = \frac{1}{n!}\int_F x_1^n\,dF +$$
$$+ \frac{b}{(n-1)!}\int_F x_1^{n-1}\,dF + \frac{b^2}{2\,(n-2)!}\int_F x_1^{n-2}\,dF +$$
$$+ \frac{b^3}{3!}\cdot\frac{1}{(n-3)!}\int_F x_1^{n-3}\,dF + \frac{b^4}{4!}\cdot\frac{1}{(n-4)!}\int_F x_1^{n-4}\,dF + \ldots +$$
$$+ \frac{b^{n-3}}{(n-3)!}\cdot\frac{1}{3!}\int_F x_1^3\,dF + \frac{b^{n-2}}{(n-2)!}\cdot\frac{1}{2}\int_F x_1^2\,dF +$$
$$+ \frac{b^{n-1}}{(n-1)!}\int_F x_1\,dF + \frac{b^n}{n!}\int_F dF = \frac{M_{1^n}}{n!} + b\,\frac{M_{1^{n-1}}}{(n-1)} +$$
$$+ \frac{b^2}{2!}\cdot\frac{M_{1^{n-2}}}{(n-2)!} + \frac{b^3}{3!}\cdot\frac{M_{1^{n-3}}}{(n-3)!} + \ldots + \frac{b^{n-3}}{(n-3)!}\cdot\frac{M_{1^3}}{3!} +$$
$$+ \frac{b^{n-2}}{(n-2)!}\cdot\frac{M_{1^2}}{2!} + \frac{b^{n-1}}{(n-1)!}\,M_{1^1} + \frac{b^n}{n!}\,M_{1^0} \tag{7.15}$$

or, in concise form *

$$\frac{M_{2^n}}{n!} = \sum_{k=n}^{k=0} \frac{M_{1^k}}{k!}\cdot\frac{b^{n-k}}{(n-k)!} \tag{7.15'}$$

Taking $n = 1, 2, 3, \ldots$ we consequently derive from (7.15) the following formulas for moments of various orders, which establish the relationship between moments with respect to two mutually parallel axes (note that $M_{1^0} = F$)

$$M_{2^1} = M_{1^1} + bF \tag{7.16}$$

$$\frac{M_{2^2}}{2!} = \frac{M_{1^2}}{2!} + bM_{1^1} + \frac{b^2}{2!}\,F \tag{7.17}$$

$$\frac{M_{2^3}}{3!} = \frac{M_{1^3}}{3!} + b\,\frac{M_{1^2}}{2!} + \frac{b^2}{2}\,M_{1^1} + \frac{b^3}{3!}\,F \tag{7.18}$$

$$\frac{M_{2^4}}{4!} = \frac{M_{1^4}}{4!} + b\,\frac{M_{1^3}}{3!} + \frac{b^2}{2!}\cdot\frac{M_{1^2}}{2!} + \frac{b^3}{3!}\,M_{1^1} + \frac{b^4}{4!}\,F \tag{7.19}$$

$$\frac{M_{2^5}}{5!} = \frac{M_{1^5}}{5!} + b\,\frac{M_{1^4}}{4!} + \frac{b^2}{2}\cdot\frac{M_{1^3}}{3!} + \frac{b^3}{3!}\cdot\frac{M_{1^2}}{2!} + \frac{b^4}{4!}\,M_{1^1} + \frac{b^5}{5!}\,F \tag{7.20}$$

---

* It should be kept in mind that $2! = 2$, $1! = 1$, $0! = 1$.

In these formulas the sign of the area itself and of its arm with respect to the moment point should be taken into account. A surface element of a distributed load, as well as of a point load, is taken positive according to the usual coordinate system (Fig. 7.3), if these loads act towards the positive direction of the ordinate axis, i.e., upwards; the arm of a surface element or of a force is considered positive if it is directed towards the positive end of the abscissa axis (to the right) when measured from the load towards the momental point (not in the opposite direction!).

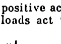

Fig. 7.3

Formulas concerning a system of point loads may be derived in a similar way by establishing the relationship between moments of such a system with respect to two different axes.

Suppose a formula is required for the relationship between moments of an m system of point loads $P_1, P_2, \ldots, P_i, \ldots, P_m$, with respect to points A and B (Fig. 7.4) placed at distances $a_1, a_2, \ldots, a_i, \ldots, a_m$ and $d_1, d_2, d_3, \ldots, d_i, \ldots, d_m$ from the forces. Then the n-th moment, divided by $n!$, with respect to point A, is determined by the following formula

$$\frac{M_{1^n}}{n!} = \sum_{i=1}^{i=m} \frac{P_i a_i^n}{n!} \tag{7.21}$$

The same forces give the following moment with respect to point B which is at a distance b from A

$$\frac{M_{2^n}}{n!} = \sum_{i=1}^{i=m} \frac{P_i d_i^n}{n!} \tag{7.22}$$

The required relationship between moments with respect to points B and A is established, if in (7.22) we substitute $d_i = b + a_i$, the formulas thus obtained are of the same type as (7.15)-(7.20), the only difference being that for a system of forces

$$M_{1^0} = M_{2^0} = \sum_{i=1}^{i=m} P_i$$

we obtain the following formula

$$\frac{M_{2^n}}{n!} = \frac{M_{1^n}}{n!} + b\frac{M_{1^{n-1}}}{(n-1)!} + \frac{b^2}{2}\cdot\frac{M_{1^{n-2}}}{(n-2)!} + \ldots +$$
$$+ \frac{b^{n-3}}{(n-3)!}\cdot\frac{M_{1^3}}{3!} + \frac{b^{n-2}}{(n-2)!}\cdot\frac{M_{1^2}}{2!} + \frac{b^{n-1}}{(n-1)!} M_{1^1} + \frac{b^n}{n!}\sum P \tag{7.23}$$

If the relationship between moments with respect to the central and arbitrary axes is to be determined, a particular case of (7.15)-

(7.20) and (7.23) appears; these formulas become simpler because the moment of the first order (statical moment) with respect to the central axis disappears; this holds for cases of combined action of distributed loads and pointed forces.

If *O-O* is the central axis (Fig. 7.5) with respect to which the statical moment (M_{O1}) of a compound load is zero, and *1-1* is the arbitrary axis at a distance c from the central axis, the following formulas may be derived on the basis of (7.15)-(7.20) and (7.23). After substituting $M_{O^0} = \sum F + \sum P$ we obtain

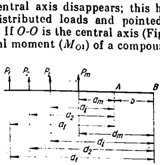

Fig. 7.4

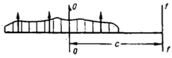

Fig. 7.5

$$M_{1^1} = cM_{O^0} \tag{7.24}$$

$$\frac{M_{1^2}}{2!} = \frac{M_{O^2}}{2!} + \frac{c^2}{2!} M_{O^0} \tag{7.25}$$

$$\frac{M_{1^3}}{3!} = \frac{M_{O^3}}{3!} + c\,\frac{M_{O^2}}{2!} + \frac{c^3}{3!} M_{O^0} \tag{7.26}$$

$$\frac{M_{1^4}}{4!} = \frac{M_{O^4}}{4!} + c\,\frac{M_{O^3}}{3!} + \frac{c^2}{2}\cdot\frac{M_{O^2}}{2!} + \frac{c^4}{4!} M_{O^0} \tag{7.27}$$

$$\frac{M_{1^5}}{5!} = \frac{M_{O^5}}{5!} + c\,\frac{M_{O^4}}{4!} + \frac{c^2}{2}\cdot\frac{M_{O^3}}{3!} + \frac{c^3}{3!}\cdot\frac{M_{O^2}}{2!} + \frac{c^5}{5!} M_{O^0} \tag{7.28}$$

Formulas for a parallel transfer of axes are simplified even more if the given load is self-balanced and consists of active, distributed or point loads and reactive factors. According to the statical equilibrium equations any self-balanced system has $M_0 = 0$ and $M_1 = 0$ with respect to any axis. That is why formulas (7.24)-(7.28), (7.15) and (7.23) acquire the following form

$$M_{1^1} = 0$$

$$\frac{M_{1^2}}{2!} = \frac{M_{O^2}}{2!} = \text{const} \tag{7.29}$$

$$\frac{M_{1^3}}{3!} = \frac{M_{O^3}}{3!} + c\,\frac{M_{O^2}}{2!} \tag{7.30}$$

$$\frac{M_{1^4}}{4!}=\frac{M_{0^4}}{4!}+c\frac{M_{0^3}}{3!}+\frac{c^2}{2}\cdot\frac{M_{0^2}}{2!} \tag{7.31}$$

$$\frac{M_{1^5}}{5!}=\frac{M_{0^5}}{5!}+c\frac{M_{0^4}}{4!}+\frac{c^2}{2!}\cdot\frac{M_{0^3}}{3!}+\frac{c^3}{3!}\cdot\frac{M_{0^2}}{2!} \tag{7.32}$$

or, in general

$$\frac{M_{1^n}}{n!}=\frac{M_{0^n}}{n!}+c\frac{M_{0^{n-1}}}{(n-1)!}+\frac{c^2}{2}\cdot\frac{M_{0^{n-2}}}{(n-2)!}+\dots+ \\ +\frac{c^{n-3}}{(n-3)!}\cdot\frac{M_{0^3}}{3!}+\frac{c^{n-2}}{(n-2)!}\ \frac{M_{0^2}}{2!} \tag{7.33}$$

Particular formulas and tables are compiled on the basis of general formulas, making it possible to determine moments of various orders not only for widely used loads (such as point loads, point force couples, distributed loads following a linear intensity law), but also for the less common types of loads seldom met (parabolic, trigonometric and all possible combinations).

The formulas derived were widely used in designing beams, frames and, to a certain extent, arches, subjected to various loadings.

8. MOMENTS OF HIGHER ORDER AND GENERALIZED FORCES

Interdependency in formula patterns for moments of gradually increasing or decreasing orders may be ascertained by examining transformations of loadings, which may be used in deriving moment formulas. It will be shown now, that one type of loading may be transformed into another on the basis of differential-integral relationships between them.

Each of the loads presented in Fig. 6.4 one under another, may be derived from the one below by a transformation which is based on the following considerations.

For example, let us examine a loading area, bounded by a parabola of the m-th order described by

$$y=A\frac{(x-x_0)^m}{m!} \quad \text{or} \quad y=B\frac{x^m}{m!}$$

Then the curve which is differential with respect to the one examined is

$$\frac{dy}{dx}=A\frac{(x-x_0)^{m-1}}{(m-1)!} \quad \text{or} \quad \frac{dy}{dx}=B\frac{x^{m-1}}{(m-1)!}$$

From this we conclude by comparing the derived equations, that each of the loads presented in Fig. 6.4 is the "differential" of the following load and the "integral" of the previous load.

Let us examine a continuous function $y = f(x)$ (Fig. 8.1.). To determine graphically the derivative of the given function along the abscissa we must show on the diagram for abscissas x and $x + \Delta x$ their respective ordinates y_1 and $y_2 = y_1 + \Delta y$; subtracting the ordinates and dividing by Δx we obtain

$$\lim \frac{y_2 - y_1}{\Delta x} = \lim \frac{\Delta y}{\Delta x} = y'$$

(if $\Delta x \to 0$); hence it follows that

$$\Delta y = y_2 - y_1 = y' \Delta x$$

Another graphical method may be used for the same purpose. Instead of giving the increment Δx to the abscissa x and thus moving from ordinate y_1 to ordinate y_2, the diagram may be moved to the left by a distance Δx so that the ordinate y_2 will be superimposed on the ordinate y_1, and the difference $\Delta y = y_2 - y_1$ may then be obtained directly.

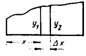

Fig. 8.1

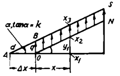

Fig. 8.2

This method may be applied to any distributed load, but it is especially useful for point forces and force couples, i.e., discontinuous functions.

This transformation will be examined further; we shall first apply it to distributed loads.

If the straight line ON passing through the origin of coordinates (point O) has the equation $y_1 = \overline{X_1X_2} = kx$ (Fig. 8.2), where $k = \tan\alpha$, then by moving this line to the left by a value $\Delta x = OA$ we obtain a new line AS with an ordinate $y_2 = \overline{X_1X_3} = k(x + \Delta x)$; the ordinate difference is $q = \Delta y = y_2 - y_1 = \overline{X_2X_3} = \overline{OB} = \tan\alpha\, \overline{OA} = k\Delta x$. Therefore, the part of the drawing containing arrows, i.e., the skewed rectangle $OBSN$, represents a vertical uniform continuous load of intensity q, which is integral with respect to k.

For the case of a uniformly distributed load of intensity q (Fig. 8.3*a*), the shift of the initial loading area $OO'B'B$ to the left by a value Δx gives a load $AA''O''B''BOA$ (Fig. 8.3*b*) of the same intensity q; subtracting the initial rectangle $OO'B'B$ from this area we obtain the rectangular area $AA''O''O$ which equals $P = q\Delta x$. Thus, between the values of q and P we find the same dependence as we formerly found for k and q; the point load is the integral of the continuous load q.

In a similar manner we may determine the relationship between the next pair of quantities (Fig. 6.4), i.e., P and M (Fig. 8.4).

Moving the force P applied at point O (Fig. 8.4a) by a distance Δx to point A on the left and subtracting the same force by an equal

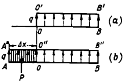

Fig. 8.3

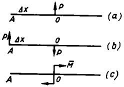

Fig. 8.4

opposite force at the initial position O (Fig. 8.4b), we obtain a force couple $\overline{M} = P\Delta x$ which may be considered acting at point O when $\Delta x \rightarrow 0$ (Fig. 8.4c).

Continuing in the same manner, let a force couple M act at point O of a beam (Fig. 8.5a). Then, according to the established rule, as in the previous case, we move the force couple $\overline{M}$ to the left by a distan-

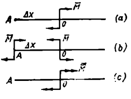

Fig. 8.5

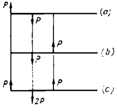

Fig. 8.6

ce Δx and place it at point A, and at point O we apply an equal and opposite couple $\overline{M}$ (Fig. 8.5b). The resulting statically balanced system of two mutually opposite force couples ($\overline{M}$) and ($-\overline{M}$) is called a force couple of the second order and designated by $\overline{\overline{M}}$ (Fig. 8.5c), where $\overline{\overline{M}} = \overline{M}\Delta x$. Substituting force couples $\overline{M}$ and $-\overline{M}$ (Fig. 8.5b) by their force diagrams shown in Figs. 8.6a and b, respectively, we obtain for the second order couple $\overline{\overline{M}}$ the force diagram shown in Fig. 8.6c.

Finally, from force couple of the second order $\overline{\overline{M}}$ we may obtain a force couple of the third order $\overline{\overline{\overline{M}}}$ (Fig. 8.7). The resulting force couple of the third order, being the sum of two opposite and statically balanced force couples of the second order, is also balanced.

A force diagram of a force couple of the third order is shown in Fig. 8.8*a* and *b*, where two force couples of the third order are replaced by appropriate forces, according to Fig. 8.6*b*. As a result the force diagram for a couple of the third order can be obtained according to Fig. 8.8*c*.

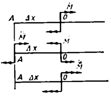

Fig. 8.7

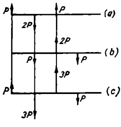

Fig. 8.8

If we continue in a similar manner, moments of the fourth (designated by $\overline{\overline{\overline{\overline{M}}}}$) and higher orders may be obtained (Fig. 8.9).

The force diagrams of force couples of the first, second and third orders, as well as higher orders, may be plotted according to Pascal's triangle (see Table 8.1), the first line of which gives 1 (for a point load), the second line —1 and 1 (for a force couple of the first order), the third line—1, 2, 1 (for a force couple of the second order), the fourth line—1, 3, 3, 1 (for a force couple of the third order), etc.

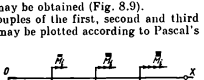
0
$\overline{M}_i$
$\overline{\overline{M}}_i$
$\overline{\overline{\overline{M}}}_i$
X
$\overline{a}_i$
$\overline{\overline{a}}_i$
$\overline{\overline{\overline{a}}}_i$
$(x-\overline{\overline{\overline{a}}}_i)$
x
$(x-\overline{a}_i)$
$(x-\overline{\overline{a}}_i)$

Fig. 8.9

Thus, the numerical coefficients for a set of parallel forces in force diagrams of force couples of any order are found in corresponding lines of Pascal's triangle. Directions of forces in the force diagrams always alternate (see the figures).

The first and the last terms in every line of Pascal's triangle are unity while any intermediate term in a line is equal to the

Table 8.1

Pascal's Triangle

Line No.																	
1									1								
2								1		1							
3							1		2		1						
4						1		3		3		1					
5					1		4		6		4		1				
6				1		5		10		10		5		1			
7			1		6		15		20		15		6		1		
8		1		7		21		35		35		21		7		1	
9	1		8		28		56		70		56		28		8		1

sum of the two terms of the previous line, which are placed directly on the right-hand and left-hand side of the terms being examined.

If the beam presented in Fig. 6.4 is supposed to bear not only the loads shown in this figure, but also statically balanced force couples of higher order, additional terms should be included in formula (I) for the n-th moment, which take into account the higher moments:

$$\sum \overline{\overline{\overline{\overline{M}}}}_i \frac{(x-\overline{\overline{\overline{\overline{a}}}}_i)^{n-4}}{(n-4)!} + \sum \overline{\overline{\overline{M}}}_i \frac{(x-\overline{\overline{\overline{a}}}_i)^{n-3}}{(n-3)!} + \sum \overline{\overline{M}}_i \frac{(x-\overline{\overline{a}}_i)^{n-2}}{(n-2)!} \qquad \text{(a)}$$

Higher couples acting upon the beam do not affect formulas (7.7)-(7.12), because these couples are in equilibrium. As already noted, the right-hand sides of the formulas gain one additional term during each step from (7.12) to (7.7). This process is observed during the derivation of the formulas for the second, third, etc. moments. This is why it is possible for the higher moments of formulas (I) and (a) to be presented in Table 8.2.

In Table 8.2 instead of the abscissa u we must substitute arms $x-a_i$, $x-b_i$, $x-c_i$, $x-d_i$, . . ., i.e., the distances to the moment axis, according to formulas (I) and (a) and Fig. 6.4.

Table 8.2

Pattern for Deriving Expressions for Moments of Arbitrary Order of Compound Loads

Order of moment	Load									
	t	s	h	q	P	$\bar{M}$	$\bar{\bar{M}}$	$\bar{\bar{\bar{M}}}$	$\bar{\bar{\bar{\bar{M}}}}$	$\frac{M_n}{n}$
-4	1	—	—	—	—	—	—	—	—	M_{-4}
-3	u	1	—	—	—	—	—	—	—	M_{-3}
-2	$\frac{u^2}{2}$	u	1	—	—	—	—	—	—	M_{-2}
-1	$\frac{u^3}{3!}$	$\frac{u^2}{2}$	u	1	—	—	—	—	—	M_{-1}
0	$\frac{u^4}{4!}$	$\frac{u^3}{3!}$	$\frac{u^2}{2}$	u	1	—	—	—	—	M_0
$+1$	$\frac{u^5}{5!}$	$\frac{u^4}{4!}$	$\frac{u^3}{3!}$	$\frac{u^2}{2}$	u	1	—	—	—	M_1
$+2$	$\frac{u^6}{6!}$	$\frac{u^5}{5!}$	$\frac{u^4}{4!}$	$\frac{u^3}{3!}$	$\frac{u^2}{2}$	u	1	—	—	$\frac{M_2}{2}$
$+3$	$\frac{u^7}{7!}$	$\frac{u^6}{6!}$	$\frac{u^5}{5!}$	$\frac{u^4}{4!}$	$\frac{u^3}{3!}$	$\frac{u^2}{2}$	u	1	—	$\frac{M_3}{3!}$

It can be seen from Table 8.2., that as we proceed from top to bottom, couples of the first order (point moments) first appear in formulas for the first moment; similarly, couples of the second order first appear in formulas for the second moment, etc.

Generalized forces repeatedly used for solving various problems concerning elasticity and resistance of materials (for example, when computing elastic media relations), irrespective of the force interpretation of the higher couples can be completely expressed by the symbolic moments of higher order. These moments are widely adopted, because they cover all possible types of loads both static (real) and "kinematic" (conditional). They are used as geometrical characteristics of sections and by the moment-operational method for designing structures and components, such as beams, frames, arches (see Chap. III and IV). Generalized forces expressed in terms of higher moments should be considered as one of the possible applications of moments in engineering practice.

Note. By integrating "kinematic" loads taking into account Table 8.2, one can see, that "a couple of the second order" (according to J. Rippenbein's terminology) should be located in the fifth line of the table, while the sixth line must be occupied by "a couple of the third order". If P. Nemenyi's terminology isto be adopted, then the third line of the table is occupied by the point load ("Einzellast als Lastgruppe erster Ordnung"); the fourth line is for the point force couple ("Einzelmoment als Lastgruppe zweiter Ordnung"); the fifth—for the load presented in Fig. 8.10 ("Doppelmoment als Lastgruppe dritter Ordnung"), and, finally, the sixth—for the load presented in Fig. 8.11 ("Doppelangriff als Lastgruppe vierter Ordnung"). Thus, all possible static and kinematic types of loads can be easily expressed by $\frac{M_n}{n!}$.

Let us examine the main principles justifying the use of continuous distributed force couples and the so-called "kinematic" loads in the general formula for the n-th moment.

Examining the right side of formula (I) we notice that force factors are arranged in a certain order: first comes the force couple, then the

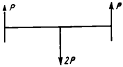

Fig. 8.10

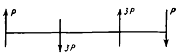

Fig. 8.11

point load, then the intensity of continuous distributed load, then the tangent of the inclination angle of the load area, the second derivative of the intensity with respect to the abscissa, etc., or, in other words, $\overline{M}_i$, P_i, q_i, k_i, s_i, t_i. These quantities form a series in which every member is differential with respect to the previous one. On the other hand, the geometrical factors contained in the terms of formula (I), are presented in the form of a series

$$\frac{(x-a_i)^{n-1}}{(n-1)!}, \quad \frac{(x-b_i)^n}{n!}, \quad \frac{(x-c_i)^{n+1}}{(n+1)!},$$
$$\frac{(x-d_i)^{n+2}}{(n+2)!}, \quad \frac{(x-e_i)^{n+3}}{(n+3)!}, \quad \frac{(x-g_i)^{n+4}}{(n+4)!}$$

in which every term is integral with respect to the previous one.

Apparently, the series in the right-hand side of formula (I) may be continued either to the left or to the right. On the right-hand side of the fourth term, the terms accounting for the parabolic loading areas are located. Just as the terms for the continuous distributed loads are located to the right of the term $\frac{P_i(x-b_i)^n}{n!}$. The terms which take into account the influence of the distributed moment loads are located to the right of $\frac{\overline{M}_i(x-a_i)^{n-1}}{(n-1)!}$.

What additional terms would appear in formula (I), if it is extrapolated to the left? When determining the elastic curve of a uniform beam it is obvious that the successive terms reflect the slope, the deflection and then the area of the elastic curve and other elements of the elastic curve multiplied by the rigidity EI of the beam.

Thus, if any section of a beam, say, the left end, has a step Θ in the inclination angle diagram at a distance s from the moment point, and a step v in the deflection diagram at a distance t from the moment point, then, according to formula (a), the expression for $\frac{M_n}{n!}$ in the right-hand side of (I) must be supplemented by two new terms

$$EI\Theta\,\frac{s^{n-2}}{(n-2)!}+EIv\,\frac{t^{n-3}}{(n-3)!} \tag{b}$$

The expression for $\frac{M_n}{n!}$ may be extended and written as a sum of the right-hand parts of (I) and (b); in this case all the initial values of the transversal force, the bending moment, the inclination angle and deflection will be automatically considered.

If distributed moments act upon a beam (Fig. 8.12), two additional terms appear in the expression for $\frac{M_n}{n!}$; they are "shifted" in Table 8.2 one and two lines higher, similar to the way the distributed loads "shifted" one and two lines higher with respect to point loads.

Thus, for the distributed moments the additional following terms are added

$$m_1\frac{(x-d_1)^n}{n!}+m_2\frac{(x-d_2)^{n+1}}{(n+1)!} \tag{c}$$

Distributed "kinematic" loads, i.e., distributed angular and linear deflections, when present, are taken into account in a similar way.

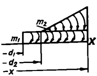

Fig. 8.12

The nature of distributed "kinematic" loads can be understood from the fact that distributed inclination angles appear when a bar is heated unevenly and its axis becomes curved. Distributed deflections are due to the action of shear forces. Introduction of distributed "kinematic" loads makes it possible to extend the moment method to these particular cases of bar deformations.

Let us assume that a beam of constant section is subjected to distributed bending moments and distributed "kinematic" loads. Let the beam be loaded with point force couples, point loads and distributed force loads as shown in Fig. 6.4. Besides, the general formula for the n-th moment should take into account initial deflections, both linear and angular, at point A with abscissa x_A.

According to (I) and (b), the n-th moment, divided by $n!$, for a beam of constant section is expressed by the following formula

$$\frac{M_n}{n!}=EIv_A\,\frac{(x-x_A)^{n-3}}{(n-3)!}+EI\Theta_A\,\frac{(x-x_A)^{n-2}}{(n-2)!}+\sum M_i\,\frac{(x-a_i)^{n-1}}{(n-1)!}+$$
$$+\sum P_i\,\frac{(x-b_i)^n}{n!}+\sum q_i\,\frac{(x-c_i)^{n+1}}{(n+1)!}+\sum k_i\,\frac{(x-d_i)^{n+2}}{(n+2)!} \tag{III}$$

The order of the moment always corresponds to the power of the term responsible for the point load in the moment series. For example, in (III), which is a formula for $\frac{M_n}{n!}$, the order of the moment corresponds to the fourth term $\sum P_i \frac{(x-b_i)^n}{n!}$.

Where experience is lacking, it is advisable to begin writing the expression for the n-th moment from the fourth term of the right-hand side; to the left of it the first three terms should be written, which take into account the point force couples, and angular and linear displacements. Then to the right from the fourth term the terms for distributed loads should be placed, such as evenly distributed and evenly varying loads and so on. Thus we complete the right-hand side of formula (III) in which the static and kinematic loads will be arranged in the following order: v, Θ, M, P, q, k, which is opposite to the order used in the graphical-analytical method of calculating strength and rigidity of beams.

All the kinematic and static quantities forming the terms of the moment series are multiplied by corresponding arms with respect to the moment point X with abscissa x. The arms are raised to powers which increase from one term to the other, i.e., $(n-3)$, $(n-2)$, $(n-1)$, n, $(n+1)$, $(n+2)$, . . ., so that the point load is multiplied by its arm raised to power n, in the general expression for the n-th moment.

Both parts of the formula are given factorial coefficients in the denominators.

It is shown in Chap. III and IV that in a correctly composed formula the higher moments and all the static and kinematic factors always have factorial denominators corresponding to the power of the numerator.

Chapter II

Application of Theory of Moments

9. RIGIDITY OF UNIFORM BEAMS

Elastic Curve of Uniform Beams

According to formula (7.1) a general expression for the transversal force Q_X and the bending moment M_X may be compiled for any loading scheme acting upon a statically determinate or a statically indeterminate beam of constant section*

$$Q_X = M_0$$

$$EIk_X = M_X = M_1 \qquad (9.1)$$

where k_X is the curvature of the beam at section X:

$$EI\Theta_X = EI\Theta_A + \frac{M_2}{2} \qquad (9.2)$$

$$EIv_X = EIv_A + EI\Theta_A x + \frac{M_3}{3!} \qquad (9.3)$$

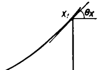

Fig. 9.1

By further integration we obtain the area of AA_1X_1X (Fig. 9.1), i.e., the area Ω_X bounded by the elastic curve, the ordinates and the abscissa (for brevity this area is called the area of the elastic curve over section AX).

By further integration we obtain S_X and I_X, i.e., the static moment and the moment of inertia of the area of the elastic curve with respect to the extreme right vertical line XX_1. All these quantities are expressed by the following formulas

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* Such symbols as M_1, M_2, . . ., only show the order of the moments and do not indicate the location of the moment points: thus they are inadequate in case higher moments with respect to various points are needed in design. In such cases symbols like M_{X2}, M_{B2}, M_{C2} should be used instead of M_2, where X, B and C are the points, with respect to which the moments of all the left-hand forces are calculated. All particular cases concerning any other loads besides the left-hand forces should be stipulated separately.

$$EI\Omega_X = EIv_A x + EI\Theta_A \frac{x^2}{2} + \frac{M_4}{4!} \tag{9.4}$$

$$EIS_X = EIv_A \frac{x^2}{2} + EI\Theta_A \frac{x^3}{3!} + \frac{M_5}{5!} \tag{9.5}$$

$$EI \frac{I_X}{2} = EIv_A \frac{x^3}{3!} + EI\Theta_A \frac{x^4}{4!} + \frac{M_6}{6!} \tag{9.6}$$

Formulas (9.2)-(9.6) may be considered particular cases of formula (III) of Sec. 8.

If the right-hand parts of formulas (9.2) and (9.3) are expanded, they result in so-called general equations for determining deflections of uniform beams

$$\begin{aligned} EI\Theta_X = {} & EI\Theta_A + \sum \overline{M}_i (x-a_i) + \sum P_i \frac{(x-b_i)^2}{2} + \\ & + \sum q_i \frac{(x-c_i)^3}{3!} + \sum k_i \frac{(x-d_i)^4}{4!} + \sum s_i \frac{(x-e_i)^5}{5!} + \\ & + \sum t_i \frac{(x-g_i)^6}{6!} \end{aligned} \tag{9.7}$$

$$\begin{aligned} EIv_X = {} & EIv_A + EI\Theta_A x + \sum \overline{M}_i \frac{(x-a_i)^2}{2} + \\ & + \sum P_i \frac{(x-b_i)^3}{3!} + \sum q_i \frac{(x-c_i)^4}{4!} + \sum k_i \frac{(x-d_i)^5}{5!} + \\ & + \sum s_i \frac{(x-e_i)^6}{6!} + \sum t_i \frac{(x-g_i)^7}{7!} \end{aligned} \tag{9.8}$$

These expressions are compiled for an arbitrary section of the beam at a distance x from the origin coordinates. All higher moments in the right-hand side of the formulas, here and further, should be determined with respect to point X.

10. GEOMETRICAL INTERPRETATION OF MOMENTS

Synthesis of Analytical and Graphical-Analytical Methods in Rigidity Design of Uniform Beams

In examining the elastic equilibrium of section AB of a beam (Fig. 10.1) we replace the interaction with parts to the left of A and to the right of B by transversal forces and bending moments. At section A we apply Q_A and M_A; at section B we apply a force opposite and equal to the transversal force (i.e., $-Q_B$) and a force couple opposite and equal to the bending moment (i.e., $-M_B$). Angular and linear deflections at the extreme sections are designated, correspondingly, by Θ_A, Θ_B, and v_A, v_B.

After all the force factors on the left of section X are taken into account, the following expressions may be derived from (9.2) and (9.3)

$$EI\Theta_X = EI\Theta_A + \frac{M_{X^2}}{2} \text{ (left forces)} \qquad (10.1)$$

and

$$EIv_X = EIv_A + EI\Theta_A x + \frac{M_{X^3}}{3!} \text{ (left forces)} \qquad (10.2)$$

The left forces include all the loads located within $[AX]$ together with Q_A and M_A, which replace the discarded part of the beam to the left of A.

If all the right force factors are to be taken into account at section X, then the following expressions may be written

$$EI\Theta_X = EI\Theta_B - \frac{M_{X^2}}{2} \text{ (right forces)} \qquad (10.3)$$

$$EIv_X = EIv_B - EI\Theta_B (e - x) - \frac{M_{X^3}}{3!} \text{ (right forces)} \qquad (10.4)$$

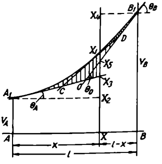

Fig. 10.1

The right forces include all the loads within BX together with Q_B and M_B, which replace the discarded part of the beam to the right of B.

The designations "left forces" and "right forces" mean that moments of the second and third order with respect to point X are taken of the forces, acting either on the left or on the right sections of the beam.

Angle Θ_O is formed between the extreme tangents A_1O and B_1O of the elastic curve at their point of intersection O; it is equal to $\Theta_O = \Theta_B - \Theta_A$. By using expressions (10.1) and (10.3) we obtain

$$EI\Theta_O = EI\Theta_B - EI\Theta_A = \frac{M_{X^2}}{2} \text{ (left forces)} + \frac{M_{X^2}}{2} \text{ (right forces)}$$

or

$$\Theta_O = \frac{M_{X^2} \text{ (all forces)}}{2EI} = \text{const} \qquad (10.5)$$

where M_{X^2} (all forces) is the moment of the second order of all the forces acting upon section AB of the beam.

Similarly, from formulas (10.2) and (10.4), we obtain

$$[v_B - \Theta_B (l - x)] - [v_A - \Theta_A x] = \frac{M_{X^3} \text{ (all forces)}}{3!\, EI} \qquad (10.6)$$

It is seen from Fig. 10.1, that $v_B = \overline{BB_1} = \overline{XX_4}$, $\Theta_B(l-x) = \overline{X_4X_5}$, $v_A = \overline{AA_1} = \overline{XX_2}$ and $\Theta_A x = \overline{X_2X_3}$, so that the left-hand side of (10.6) determines the length of $\overline{X_3X_5}$, where X_3 and X_5 are points of intersection of the vertical line (with abscissa x) with the two extreme tangents of the elastic curve, which pass through points A and B.

By expanding formula (9.4) for the left and the right forces we obtain

$$EI\Omega_{A-X} = EIv_A x + EI\Theta_A \frac{x^2}{2} + \frac{M_{X^4}}{4!} \text{ (left forces)} \qquad (10.7)$$

and

$$EI\Omega_{B-X} = -EIv_B(l-x) + EI\Theta_B \frac{(l-x)^2}{2} - \frac{M_{X^4}}{4!} \text{ (right forces)} \quad (10.8)$$

Hence,

$$EI\Omega_{X-B} = -EI\Omega_{B-X} = EIv_B(l-x) -$$
$$- EI\Theta_B \frac{(l-x)^2}{2} + \frac{M_{X^4}}{4!} \text{ (right forces)} \qquad (10.8a)$$

By adding formulas (10.7) and (10.8*a*) term-by-term we find

$$\Omega_X = \left[v_A x + \Theta_A \frac{x^2}{2}\right] + \left[v_B(l-x) - \Theta_B \frac{(l-x)^2}{2}\right] + \frac{M_{X^4}}{4!} \text{ (all forces)} \qquad (10.9)$$

These calculations take into account the fact that Ω_{A-X} is the area of figure AA_1X_1X; Ω_{X-B} is the area of XX_1B_1B; Ω is the area of $AA_1X_1B_1B$; v_Ax is the area of AA_1X_2X; $\Theta_A \frac{x^2}{2}$ is the area of $A_1X_3X_2$; $v_B\,(l-x)$ is the area of XX_4B_1B; $\Theta_B\ \frac{(l-x)^2}{2}$ is the area of $B_1X_4X_5$; and consequently, the fourth moment $\frac{MX^4}{4!}$ (all forces) is equal to the area of $A_1X_1B_1X_5X_3A_1$ (shaded in Fig. 10.1), and so on.

However, since the angles of inclination at the bearings Θ_A and Θ_B are determined by the graphical-analytical method as reactions from a fictitious load, the angle Θ_O, according to Fig. 10.1, multiplied by the rigidity of the beam EI equals the sum of these reactions. The latter, in its turn, is equal to the fictitious load, i.e. to the area of the bending moments diagram.

If a funicular polygon is plotted for the fictitious loads acting upon the beam, and the polar distance is equal to EI, then the funicular polygon coincides with the elastic curve $A_1X_1B_1$, and it may be used for determining the ordinates by measuring them along the verticals. Static moments of the fictitious loads are obtained when these ordinates are multiplied by the polar distance EI.

According to the established rules, when determining moments of inertia of a figure by the graphical method, the area bounded by the funicular polygon (i.e., the elastic curve), the extreme tangents and the moment vertical line multiplied by the polar distance gives the value of the second moment (divided by 2!) of the moment diagram load.

On the basis of the previous discussion the following theorems may be formulated. They are of obvious theoretical interest and of great practical importance for designing beams of constant rigidity and are valid for the entire beam or any part of it.

Theorem IX. *Half of the second moment of all vertical (both active and reactive) balanced forces with respect to the axis which passes through an arbitrary point, is a constant value equal to the following quantities:* (1) *the first (static) moment of the area of the transversal force diagram;* (2) *the zero moment, i.e., the area of the bending moment diagram;* (3) *the angle* Θ_0 *between the extreme tangents of the elastic curve (these tangents may be drawn from the ends of the entire beam or the part examined), multiplied by the rigidity* EI *of the beam*

$$\frac{M_{X^2}}{2}\ (\text{all forces}) = M_{X^1}\ (Q\ \text{diagram}) = M_{X^0}\ (M\ \text{diagram}) = EI\Theta_0 \tag{10.10}$$

Theorem X. *The third moment (divided by 3!) of all vertical balanced forces with respect to the axis which passes through an arbitrary point is a linear function in terms of the abscissa of the moment point. The value of* $\frac{M_3}{3!}$ *is numerically equal to the following quantities:* (1) *the second moment (divided by 2!) of the transversal force diagram with respect to the vertical line passing through the moment point;* (2) *the first (static) moment of the bending moment diagram with respect to the same vertical line;* (3) *the ordinate between the two extreme tangents of the elastic curve, measured along the vertical line*, multiplied by* EI

$$\frac{M_{X^3}}{3!}\ (\text{all forces}) = \frac{M_{X^2}}{2}\ (Q\ \text{diagram}) = M_{X^1}\ (M\ \text{diagram}) = EIy \tag{10.11}$$

Theorem XI. *The fourth moment (divided by 4!) of all vertical balanced forces with respect to the axis passing through an arbitrary point** is a second order parabolic function of the abscissa of the moment*

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* The ordinate is considered positive if it is directed towards the positive end of the ordinate axis, i.e., upwards, when measured along the vertical line from its point of intersection with the extreme left tangent to that of the extreme right one; the ordinate $y=\overline{X_3X_5}$ in Fig. 10.1 is positive.

** The point X and the moment vertical line can be located beyond the span, i.e., at any point of the extrapolated axis of the beam.

point. The value of $\frac{M_4}{4!}$ *is numerically equal to the following quantities: (1) the third moment (divided by 3!) of the transverse force diagram with respect to the vertical line passing through the moment point; (2) the second moment (divided by 2!) of the bending moment diagram with respect to the same vertical line; (3) the area* Ω *of the figure bounded by the elastic curve, the extreme tangents and the moment vertical line, multiplied by EI*

$$\frac{M_{X^4}}{4!}\text{ (all forces)} = \frac{M_{X^3}}{3!}\ (Q\text{ diagram}) = \frac{M_{X^2}}{2}\ (M\text{ diagram}) = EI\Omega \tag{10.12}$$

Theorem XII. *The fifth moment (divided by 5!) of all vertical balanced forces with respect to the axis passing through an arbitrary point is a third order parabolic function of the abscissa of the moment point. The value of* $\frac{M_5}{5!}$ *is numerically equal to the following quantities: (1) the fourth moment (divided by 4!) of the transversal force diagram with respect to the vertical line passing through point* X; *(2) the third moment (divided by 3!) of the bending moment diagram with respect to the same vertical line; (3) the static moment of the figure bounded by the elastic curve, the extreme tangents and the moment vertical line, with respect to the same vertical line, multiplied by EI*

$$\frac{M_{X^5}}{5!}\text{ (all forces)} = \frac{M_{X^4}}{4!}\ (Q\text{ diagram}) = \frac{M_{X^3}}{3!}\ (M\text{ diagram}) = EIS_X \tag{10.13}$$

Theorem XIII. *The sixth moment (divided by 6!) of all vertical balanced forces with respect to the axis passing through an arbitrary point is a fourth order parabolic function of the abscissa of the moment point. The value of* $\frac{M_6}{6!}$ *is numerically equal to the following quantities: (1) the fifth moment (divided by 5!) of the transversal force diagram with respect to the moment vertical line; (2) the fourth moment (divided by 4!) of the bending moment diagram with respect to the same vertical line; (3) the second moment (divided by 2!) of the figure bounded by the elastic curve, the extreme tangents and the moment vertical with respect to the same vertical line, multiplied by EI*

$$\frac{M_{X^6}}{6!}\text{ (all forces)} = \frac{M_{X^5}}{5!}\ (Q\text{ diagram}) =$$

$$= \frac{M_{X^4}}{4!}\ (M\text{ diagram}) = EI\,\frac{M_2}{2}\,(\Omega) \tag{10.14}$$

These theorems are of a general character and include, as particular cases, the graphical-analytical method of designing beams and the general equation of the elastic curve*.

The well-known Vereshchagin's rule for calculating displacement integrals is a direct consequence of these theorems which are of great significance. Besides their broad theoretical generalization, they make it possible to solve easily a number of engineering problems which were more difficult to solve by other methods.

It was proved in Sec. 4, that if the force characteristics are arranged in the following order: $\overline{\overline{\overline{\overline{M}}}}$ (a fourth order couple), $\overline{\overline{\overline{M}}}$ (a third order couple), $\overline{\overline{M}}$ (a second order couple), $\overline{M}$ (a first order couple), P (a point load), q (intensity of an evenly distributed load), k (tangent of an inclination angle of an inclined loading line), s and t (parameters of the second and third order parabolic equations),—then every quantity is differential with respect to the previous one and integral with respect to the following one.

Let us illustrate this by an example.

Let a beam be fixed at the right end and loaded by an evenly distributed load acting upon the section $BC = b$, while the left part $AB = a$ is free of any loading (Fig. 10.2).

If the left end A of the beam is also fixed, then in designing an unknown point load and point force couple should be applied to the extreme left section of the beam; these unknown quantities should be equal to the reactive factors in this section. In this first case the left end of the beam is considered free.

If all the quantities are examined from the standpoint of reduced loads, then we have a series of loads with integral-differential relationships between the members.

From a rectangular loading area (Fig. 10.2*d*) a consequence of integral diagrams (Q, M, $EI\Theta$, EIv) follows in the form of a triangle (Fig. 10.2*e*), a second order parabola (Fig. 10.2*f*), a third order parabola (Fig. 10.2*g*), a fourth order parabola (Fig. 10.2*h*). If we proceed in the opposite direction, from the loading area (Fig. 10.2*d*) then a consequence of differential diagrams (k, s, t) follows in the form of a point load (Fig. 10.2*c*), force couple of the first order (Fig. 10.2*b*) and a force couple of the second order (Fig. 10.2*a*), all these reduced loads being numerically equal to q.

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* The general equation of the elastic curve is called by some authors the "universal" formula or the "universal" equation. This is hardly correct, because this formula can be applied only when a number of conditions is satisfied: all sections are equal, all forces are vertical, no transversal forces influence the deflections of the beam, there are no temperature influences, the supports are absolutely rigid, all deflections are small, there are no lateral forces and no other loads except the ones presented by the algebraic polynomial.

In a similar way, "kinematic" factors, such as the initial deflection EIv_A and the initial inclination angle $EI\Theta_A$, which appear at the free left end of the beam must be taken into account. By using reduced loads, the initial deflection being multiplied by the rigidity of the beam must be represented on the previous differential diagram $EI\Theta$ by a point load (Fig. 10.2*g*). This will result in a rectangle add_3a_1 with an ordinate $aa_1 = dd_3 = EIv_A$ in the EIv diagram, which is integral with respect to the $EI\Theta$ diagram (Fig. 10.2*h*). According to the sequence of reduced loads established above, a point load EIv_A on the inclination angle diagram (Fig. 10.2*g*) will undergo the following transformations in the preceding diagrams: a force couple (Fig. 10.2*f*), a force couple of the second order (Fig. 10.2*e*), a force couple of the third order (Fig. 10.2*d*), and so on, up to a force couple of the sixth order on the *t* diagram (Fig. 10.2*a*).

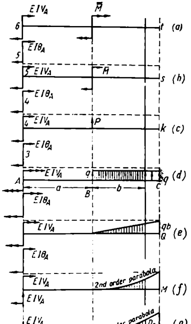

Fig. 10.2

The initial inclination angle $EI\Theta_A$ must be expressed by a point force on the bending moment diagram (Fig. 10.2*f*) which precedes the integral inclination angle diagram. According to the assumed sequence, this point load on the bending moment diagram (Fig. 10.2*f*) will be consequently transformed on the differential diagrams, which precede the bending moment diagram on the scheme, into force couples of the first, second, ... order, up to a force couple of the fifth order on the *t* diagram (Fig. 10.2*a*). As for the integral diagrams which follow the point load

(Fig. 10.2*f*), a rectangle appears on the angle diagram (Fig. 10.2*g*) and a triangle, on the deflection diagram (Fig. 10.2*h*).

If the left end of the beam is fixed, then a point load (reaction A) and a force couple (support moment M_A) must be shown on the distributed load diagram q (Fig. 10.2*a*) at the left section. All other diagrams, both above and below the loading diagram of these force factors, will show reduced loads according to the sequence previously established.

Let us examine these load transformations analytically.

Suppose I_m is an m-multiple integral of function $f(x)$

$$I_m = \int\int \dots \int f(x)\,dx^m \qquad (10.15)$$

Since all the quantities presented schematically in Fig. 10.2 have differential-integral relationships, each one may be derived from any other by a certain number of integrations. For example, the deflection (multiplied by the rigidity of the beam) may be expressed as

$$\begin{aligned} EIv &= I_1(EI\Theta) = I_2(M) = I_3(Q) = I_4(q) = I_5(k) = \\ &= I_6(s) = I_7(t) \end{aligned} \qquad (10.16)$$

where I_1, I_2, . . . are the first, second, ... integrals, respectively.

According to the above mentioned relationship between multiple integrals and higher moments [see formula (II), Sec. 7], formula (10.16) may be transformed by substituting higher moments instead of multiple integrals. This results in the following formula

$$\begin{aligned} EIv &= M_0(EI\Theta) = M_1(M) = \frac{M_2}{2}(Q) = \\ &= \frac{M_3}{3!}(q) = \frac{M_4}{4!}(k) = \frac{M_5}{5!}(s) = \frac{M_6}{6!}(t) \end{aligned} \qquad (10.17)$$

Formulas (10.11) and (10.17) are essentially the same. The only difference is that when formula (10.11) and theorems IX-XIII are used, appropriate moments must be calculated for the total load on the left- and on the right-hand sides of the moment axis, as well as from the diagrams both on the left- and on the right-hand sides of the momental axis. When formulas (10.10)-(10.14) are used, the "kinematic" factors (i.e., the boundary values of the angular and linear displacements) are not included in the formulas; as a result, angles, ordinates, areas, etc. expressed by the last terms of these formulas correspond to the shaded part of Fig. 10.1.

Theorems IX-XIII may be applied not only to section AB of the beam located on both sides of the X vertical line (Fig. 10.1).

but to the left section AX or to the right section BX as well. If only the left part is analysed, then points A and X will be the boundary points and only the A_1X_1 part of the elastic curve will be examined.

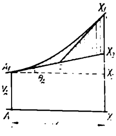

Fig. 10.3

After the right part is rejected, thus leaving the left part AX for examination (Fig. 10.3), the straight lines A_1CX_3 and X_1C become the extreme tangents and all the conclusions drawn for the total section AB and the shaded area in Fig. 10.1 will now apply to the left part AX of section AB and to the shaded area shown in Fig. 10.3.

The results obtained can be reduced to the following theorem.

Theorem XIV. *If any part of a beam taken between two cross sections is balanced by boundary force factors (i.e., forces and force couples) on the left- and on the right-hand sides and an arbitrary section is examined within this part, then for an expression* $\frac{M_n}{n!}$ *determined with respect to this section we have**

$$\frac{M_n}{n!}\ (EIv\ \text{diagram}) = \frac{M_{n+1}}{(n+1)!}\ (EI\Theta\ \text{diagram}) =$$
$$= \frac{M_{n+2}}{(n+2)!}\ (M\ \text{diagram}) = \frac{M_{n+3}}{(n+3)!}\ (Q\ \text{diagram}) =$$
$$= \frac{M_{n+4}}{(n+4)!}\ (q\ \text{diagram}) = \frac{M_{n+5}}{(n+5)!}\ (k\ \text{diagram}) = \ \dots \qquad (10.18)$$

This formula may be continued to the left by adding terms which contain, according to the previously established rule, moments of successively decreasing order (n—1, n—2, ...) for the elastic curve area, for the static moment of the elastic curve area, for its moment of inertia (divided by 2), and so on.

According to (10.18), the value of $\frac{M_n}{n!}$ for the deflection diagram, multiplied by EI, may be determined in several ways because it is equal to

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* (a) These relations may be written both with the boundary "kinematic" factors taken into account (Figs. 10.1 and 10.3) and without them (Fig. 10.2); in the first case the shaded areas are considered, in the second—the areas bounded by the elastic curve and the abscissa;

(b) Formula (10.17) may be derived from this theorem, if we assume $n = 1$. It follows from (10.17), that the deflection ordinate (multiplied by EI) may be obtained by determining the areas of deviation diagrams (multiplied by EI) in several ways: first, by determining the static moments of the bending moment diagram (graphical-analytical method); second, by determining moments of inertia (divided by 2) of the transversal force diagram; third, by determining third moments (divided by 3!) of the initial load.

(1) $\frac{M_{n+1}}{(n+1)!}$ of the deflection diagram, multiplied by EI

(2) $\frac{M_{n+2}}{(n+2)!}$ of the bending moment diagram M

(3) $\frac{M_{n+3}}{(n+3)!}$ of the transversal force diagram Q

(4) $\frac{M_{n+4}}{(n+4)!}$ of the uniformly distributed load intensity diagram q

(5) $\frac{M_{n+5}}{(n+5)!}$ of the uniformly varying load intensity diagram k

All these diagrams are examined within the selected part of the beam. In a particular case, all the relations hold for the entire beam as well.

11. CALCULATION OF DISPLACEMENT INTEGRALS IN ROD SYSTEMS

Theorems and formulas of Sec. 6 may be applied for calculating displacement integrals in rod systems with members of constant rigidity.

Let one of the quantities, namely M, which is a member of the following displacement integral

$$I=\int \frac{\overline{M}M}{EI}\,du \tag{11.1}$$

be expressed by ordinates of an arbitrary shaped diagram (Fig. 11.1*a*) and let the ordinates of the other function $\overline{M}$, which is "multiplied" by M, vary in different cases according to different laws; some of these cases will be considered below.

(1) If $\overline{M}=M_1=\text{const}$, then

$$I_1=\frac{M_1}{EI}\int M\,du=\frac{1}{EI}M_1\Omega(M) \tag{11.2}$$

where $\Omega(M)$ is the area of the moment diagram M (Fig. 11.1*b*).

According to formula (10.10), the area of the moment diagram may be calculated without constructing the diagram as the moment $\frac{M_2}{2}$, i.e., directly as a function of the load (including the reactions) which gives the moment diagram M.

Thus,

$$I_1=\frac{1}{EI}M_1\frac{M_2}{2} \tag{11.3}$$

After we designate $M_1=h_1$ and combine formulas (11.2) and (11.3), according to (10.10) we obtain the following relation

$$I_1=\frac{h_1}{EI}\cdot\frac{M_2}{2}=\frac{h_1}{EI}\Omega(M)=h_1\Theta_O \tag{11.4}$$

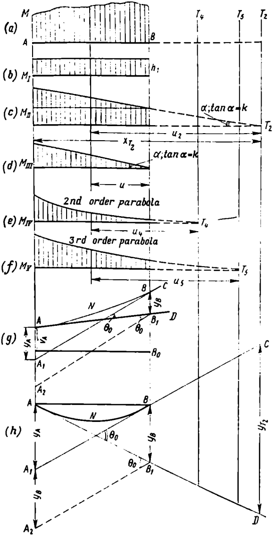

M
(a)
A
B
T₄
T₅
T₂
(b) M_I
h₁
(c) M_II
α; tan α = k
T₂
u₂
x_T₂
(d) M_III
α; tan α = k
u
2nd order parabola
(e) M_IV
T₄
u₄
3rd order parabola
(f) M_V
T₅
u₅
B
C
N
y_B
A
D
θ₀
θ₀
B₁
(g)
y_A
y_A
A₁
B₀
A₂
A
B
N
(h)
y_A
θ₀
y_B
y_T₂
C
A₁
θ₀
B₁
y_B
A₂
D

Fig. 11.1

because the area of the moment diagram, $\Omega(M)$, defines the inclination angle Θ_O (Fig. 11.1*g*, *h*)*.

(2) If the ordinates of the diagram $\overline{M} = M_{II}$ follow a linear law with a "vanishing" point T_2, then (Fig. 11.1*c*)

$$M_{II} = ku_2$$

and

$$I_{II} = \int \frac{M_{II}M}{EI}\,du_2 = \frac{k}{EI}\int u_2M\,du_2$$

However, $M\,du_2$ is area element of diagram M, $u_2M\,du_2$ is the static moment of this area element with respect to the T_2-T_2 axis, and $\int u_2M\,du_2$ is the static moment of the total diagram M with respect to the same axis.

If straight lines AB_1D and A_1BC are the extreme tangents of the elastic curve and y_B and y_A are sections of the support vertical lines cut off by these tangents which intersect at an angle Θ_O, then, according to formula (10.11) and Theorem X, we have

$$I_{II} = \frac{k}{EI}\cdot\frac{M_{T_2^3}}{3!} = \frac{k}{EI}M_{T_2^1}\ (M\text{ diagram}) = ky_{T_2} \qquad (11.5)$$

In a particular case, when the "vanishing" point of a linear diagram s on the support vertical line (Fig. 11.1*h*) or, in other words, if the moment M_{III} diagram is a triangle, then formula (11.5) transforms into the following

$$I_{III} = \frac{k}{EI}\cdot\frac{M_{B^3}}{3!} = \frac{k}{EI}M_{B^1}\ (M\text{ diagram}) = ky_B \qquad (11.6)$$

Segment y_B is shown in Fig. 11.1*g* and *h*. If a straight line $A_2B_1 \parallel A_1B$ is drawn, then from triangle AB_1A_2 we find

$$\overline{AA_2} = y_A + y_B = \Theta_O l$$

hence

$$y_A = \Theta_O l - y_B \qquad (11.7)$$

where $l = AB$.

After calculating**

$$\Theta_O = \frac{1}{EI}\cdot\frac{M_{B^2}}{2!}$$

---

* The elastic curve in Fig. 11.1*g* is drawn for the general case; it is supposed to follow the curve ANB with positive boundary deflection ordinates $\overline{A_0A} = v$, $\overline{B_0B} = v_B$; the elastic curve in Fig. 11.1*h* is supposed to follow the curve ANB with zero deflections at points A and B, which is true for a free beam resting on two supports.

** It was proved previously, that the moment M_2 of the second order of a balanced load is a constant value; however, in (11.7) and (11.8) it is more convenient to calculate both moments M_{B2} and M_{B3} with respect to point B.

and

$$y_B = \frac{1}{EI} \cdot \frac{M_{B^2}}{3!} \tag{11.8}$$

we determine y_A from formula (11.7). After plotting y_A and y_B on the support verticals we draw the extreme tangents AB_1D and A_1BC for the elastic curve (Fig. 11.1*h*). Now y_{T_2}, as well as any other segment between the extreme tangents, may be obtained graphically, from the drawing, or analytically.

If x_{T_2} is the abscissa of point T_2 when measured from point A, then

$$y_A + y_{T_2} = \Theta_O x_{T_2}$$

hence

$$y_{T_2} = \Theta_O x_{T_2} - y_A \tag{11.9}$$

According to Theorems XI and XII, displacement integrals of diagrams representing parabolas of second and third order may be calculated in a similar way (Fig. 11.1*e* and *f*).

(3) If $\overline{M} = M_{IV} = \frac{pu_4^2}{2}$, then

$$I_{IV} = \int \frac{M_{IV}M}{EI} du_4 = \frac{p}{2EI} \int (M\, du_4)\, u_4^2 = \frac{p}{EI} \cdot \frac{M_{T_4^3}}{2} \text{ (}M\text{ diagram)}$$

and, finally,

$$I_{IV} = \frac{p}{EI} \cdot \frac{M_{T_4^4}}{4!} = \frac{p}{EI} \cdot \frac{M_{T_4^2}}{2} \text{ (}M\text{ diagram)} \tag{11.10}$$

(4) If $\overline{M} = M_V = \frac{r}{3!} u_5^3$, then

$$I_V = \int \frac{M_V M}{EI} du_5 = \frac{r}{3!\ EI} \int (M\, du_5)\, u_5^3$$

and, finally,

$$I_V = \frac{r}{EI} \cdot \frac{M_{T_5^3} \text{ (}M\text{ diagram)}}{3!} = \frac{r}{EI} \cdot \frac{M_{T_5^5}}{5!} \tag{11.11}$$

(5) If the moment diagram is a triangle with an ordinate h at distances a and b from the supports (Fig. 11.2), then the straight line BC_1 may be continued until it meets the left vertical line at point A_1, and the given diagram may be thus replaced by two triangles aba_1 and $a_0c_0a_2$ of opposite signs. In this case the displacement integral will be expressed by the following formula, according to (11.5),

$$I = \frac{k_1}{EI} \cdot \frac{M_{B^3}}{3!} - \frac{k_2}{EI} \cdot \frac{M_{C^3}}{3!} \tag{11.12}$$

where M_{B^3} and M_{C^3} are third moments of the forces (active and reactive) lying to the left of points B and C; $k_1 = \tan\alpha$ and $k_2 = \tan\beta$.

(б) If one of the "multiplied" diagrams is a parabola, which is symmetrical about the centre vertical (Fig. 11.3*a*), its largest ordinate being equal to h, then the displacement integral should be calculated by the following formula

$$I = \int \frac{M\overline{M}}{EI}\,du$$

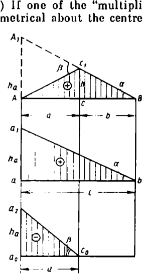


Fig. 11.2

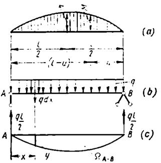


Fig. 11.3

with ordinates of the symmetrical parabola following the law

$$\overline{M} = \frac{4h}{l^2}\,u\,(l-u)$$

Hence

$$I = \frac{1}{EI}\cdot\frac{4h}{l^2}\left[l\int M\cdot du\cdot u - 2\int M\cdot du\,\frac{u^2}{2}\right] =$$

$$= \frac{1}{EI}\cdot\frac{4h}{l^2}\left[lM_{B^1}\,(M\text{ diagram}) - 2\,\frac{M_{B^2}\,(M\text{ diagram})}{2!}\right] =$$

$$= \frac{1}{EI}\cdot\frac{4h}{l^2}\left[l\,\frac{M_{B^3}}{3!} - 2\,\frac{M_{B^4}}{4!}\right] \tag{11.13}$$

According to Theorem XIV, the first and second moments of the moment diagram are replaced in this formula by the third and fourth moments of the load.

The moment diagram $\overline{M}$ (Fig. 11.3*a*) obtained for a free beam resting on two supports with a load uniformly distributed over the total span (Fig. 11.3*б*) has a maximum bending moment $h = \frac{ql^2}{8}$, hence $q = \frac{8h}{l^2}$. The work of such a load along the displacements caused by different system of forces (Fig. 11.3*c*) is equal to the sum of works produced along these displacements by each element $q \cdot dx$ of the load area

$$-\int q \cdot dx \cdot v = -q \int v \cdot dx = -q\Omega_{A-B} \tag{a}$$

where Ω_{A-B} is the area bounded by the elastic curve over the total span.

From the equations of displacements

$$\Theta_X = \Theta_A + \frac{M_{X^2}}{2EI}, \quad v_X = \Theta_A x + \frac{M_{X^3}}{3!\ EI}$$

we obtain

$$\Omega_X = \Theta_A \frac{x^2}{2} + \frac{M_{X^4}}{4!\ EI}$$

and, substituting $x = l$ into the last equation for the area Ω_X of elastic curve, we find

$$EI\Omega_{A-B} = -\frac{M_{B^3}}{3!} \cdot \frac{l}{2} + \frac{M_{B^4}}{4!} \tag{b}$$

since

$$\Theta_A = -\frac{M_{B^3}}{3!} \cdot \frac{1}{lEI}$$

Substituting Ω_{A-B} from (b) into (a), we again arrive at (11.13).

(7) If one of the "multiplied" diagrams consists of two triangles of opposite signs (Fig. 11.4), it may be replaced by two diagrams: a positive triangle with ordinate $(h_1 + h_2)$ and "vanishing" point at the right support, and a negative rectangle with ordinate h_2. Then formula (11.9) may be used.

On the other hand, we can write directly

$$I = \int \frac{M\overline{M}}{EI} dx = \frac{h_1 + h_2}{l} \cdot \frac{M_{O^3}}{3!} \tag{11.14}$$

since the "vanishing" point is in O.

When point O is in the middle of the span, then diagram M is asymmetrical, which means that $I = 0$ under symmetrical load because $M_{O^3} = 0$, according to Theorem VII*.

---

* Besides the cases examined in this section, many non-linear diagrams are involved in engineering practice, for which formulas may be compiled in a similar manner.

Problem 1. Determine displacement integrals for a beam AB (Fig. 11.5) where the first "multiplied" diagram corresponds to one of the curves in Fig. 11.5*b-f*, and the second diagram corresponds to the load shown in Fig. 11.5*a*. The latter need not be built, because all the results may be obtained directly from the given load by means of formulas (11.4)-(11.14).

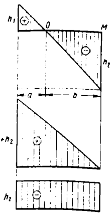


Fig. 11.4

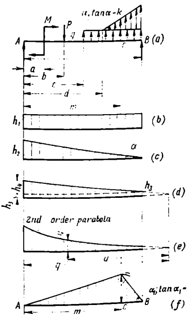


Fig. 11.5

When $n = 2$, according to formula (I), Sec. 6, we find

$$\frac{M_{B^2}}{2EI} = \left\{ \overline{M}(l-a) + \left[A\frac{l^2}{2} - P\frac{(l-b)^2}{2} \right] + \right.$$
$$\left. + q\frac{(l-c)^3}{3!} + \overline{k}\frac{(l-d)^4}{4!} \right\} \frac{1}{EI} = b_2 \tag{11.15}$$

when $n = 3$

$$\frac{M_{B^3}}{3!\ EI} = \left\{ \overline{M}\frac{(l-a)^2}{2} + \left[A\frac{l^3}{3!} - P\frac{(l-b)^3}{3!} \right] + \right.$$
$$\left. + q\frac{(l-c)^4}{4!} + \overline{k}\frac{(l-d)^6}{5!} \right\} \frac{1}{EI} = b_3 \tag{11.16}$$

If the moment diagram for a unit load is a rectangle of height h_1 (Fig. 11.5*b*), then

$$I = h_1 b_2 \tag{11.17}$$

If the moment diagram is a triangle (Fig. 11.5*c*) with $\bar{k} = \tan\alpha = \frac{h_2}{l}$, then

$$I = kb_3 \tag{11.18}$$

If the diagram is a trapezium (Fig. 11.5*d*) with ordinates $(h_3 + h_4)$ and h_3 at the supports, then it may be divided into a rectangle with an ordinate h_3 and a triangle with $\bar{k} = \frac{h_4}{l}$. Then the required result may be obtained as a sum of two terms

$$I = h_3 b_2 + \bar{k} b_3 \tag{11.19}$$

If the ordinates of a diagram are expressed by a parabola $y = \frac{pu^2}{2}$ with its apex at point D at distance g from the origin of coordinates,

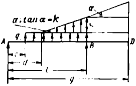


Fig. 11.6

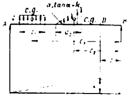


Fig. 11.7

then the integral may be calculated by formula (11.10) for the diagram in Fig. 11.5*e*. Since this formula contains the fourth moment of the load with respect to an axis passing through point D, the distributed load may be continued to the moment axis (Fig. 11.6) and the additional areas shown by dotted lines subtracted for calculation conveniency. By designating the B-support ordinate as $r = k(l - d)$ and using formula (11.10) we derive

$$I = \frac{p}{EI} \cdot \frac{M_{D^4}}{4!} = \frac{p}{EI} \left\{ \bar{M} \frac{(g-a)^3}{3!} + \left[A \frac{g^4}{4!} - P \frac{(g-b)^4}{4!} + B \frac{(g-l)^4}{4!} \right] + \right.$$
$$\left. + \frac{q(g-c)^5}{5!} + k \frac{(g-d)^6}{6!} - (q+r) \frac{(g-l)^5}{5!} - k \frac{(g-l)^6}{6!} \right\} \tag{11.20}$$

Finally, for a triangular diagram (Fig. 11.5*f*), according to (11.12) and (11.16) the following formula is derived

$$I + k_1 b_3 - \frac{k_2}{EI} \left\{ \bar{M} \frac{(m-a)^2}{2} + \left[A \frac{m^3}{3!} - P \frac{(m-b)^3}{3!} \right] + \right.$$
$$\left. + q \frac{(m-c)^4}{4!} + k \frac{(m-d)^5}{5!} \right\} \tag{11.21}$$

Problem 2. Determine displacements $\left(\text{by integrating } \int \frac{M_1 M}{EI}\, dx\right)$ for two distributed loads applied to a beam (Fig. 11.7): (1) with uniform intensity q over span a_1 and (2) with uniformly increasing intensity expressed by a triangular loading area of length a_2 and an angular coefficient k. Let the moment diagram of a unit load be a triangle with an ordinate h at point D at distance d from support B.

The distances from the right support B to centres of gravity (c.g.) of the rectangle and the triangle are designated, correspondingly, by c_1 and c_2; the angular coefficients of diagram $\overline{M}$ are $k_1 = \tan\alpha_1 = \frac{h}{d}$ and $k_2 = \frac{k_1 l}{l-d} = \frac{hl}{d(l-d)}$.

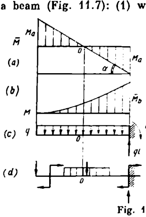


Fig. 11.8

By applying formula (5.3) and Figs. 5.4 and 5.7, the following expression is derived according to formula (11.12)

$$I = \frac{k_1}{EI}\left\{\frac{Al^3}{3!} + \left[O + c_1\left(+\frac{qa_1^3}{24}\right) + \frac{c_1^3}{3!}(+ga_1)\right] + \right.$$
$$+\left[+\frac{ka_2^5}{1620} + c^2\left(+\frac{ka_2^4}{72}\right) + \frac{c_2^3}{3!}\left(+\frac{ka_2^2}{2}\right)\right]\Bigg\} -$$
$$-\frac{k_2}{EI}\left\{A\,\frac{(l-d)^3}{3!} + \left[O + (c_1-d)\left(+\frac{qa_1^3}{24}\right) + \frac{(c_1-d)^3}{3!}(+qa_1)\right] + \right.$$
$$\left. +\left[+\frac{ka_2^5}{1620} + (c_2-d)\left(+\frac{ka_2^4}{72}\right) + \frac{(c_2-d)^3}{3!}\left(+\frac{ka_2^2}{2}\right)\right]\right\} \qquad (11.22)^*$$

Problem 3. Let the diagram $\overline{M}$ be asymmetrical (Fig. 11.8*a*) and the diagram M is a second order parabola with a maximum ordinate $\overline{M}_b$ (Fig. 11.8*b*). Diagram M may be considered a moment diagram of a uniformly distributed load of intensity q, which acts upon a beam with a fixed right end (Fig. 11.8*c*), and $\overline{M}_b = \frac{ql^2}{2}$. By determining the reactive third moment $B = ql$ and the support moment $\overline{M}_b = \frac{ql^2}{2}$ with respect to the axis passing through point O,

---

[*] The first brackets in both terms contain expressions for the third moments (divided by 3!) of the rectangle, the second brackets are the same expressions for the triangle. Both distributed loads are taken positive, because they tend towards the positive direction of the ordinate axis.

from formula (11.14) we find

$$I = -\frac{2M_A}{l}\left[-ql\frac{\left(\frac{l}{2}\right)^3}{3!} + \frac{ql^2}{2}\cdot\frac{\left(\frac{l}{2}\right)^2}{2!}\right] = -\frac{M_a\bar{M}_b l}{6}$$

since the third moment of the rectangle area is zero and

$$\tan\alpha = \frac{2M_a}{l} \quad \text{and} \quad q = \frac{2M_b}{l^2}$$

If a compound load acts upon the beam (Fig. 11.8*d*) and the moment diagram $\bar{M}$ is the same as before, then the displacement integral should be determined by calculating the third moment of the load and the reactions with respect to point O (Fig. 11.8*a*).

12. APPLICATION OF HIGHER MOMENTS TO LOADING OF PARABOLIC INFLUENCE LINES

Formulas compiled in terms of higher moments consider both the diagrams and the influence lines. In the previous discussion we examined certain loadings and derived rules for determining moments of various loads [formulas (I)-(III)], according to the symbols of moments. The influence line equation is derived from such formulas by substituting the values of the abscissa raised to the zero, first, second, third, etc. powers, instead of the following moments: M_0, M_1, M_2, M_3, etc. The power of the abscissa corresponds to the order of the moment.

Regardless of the way of obtaining the influence line equation (whether it is derived from a moment expression or by any other, non-moment method), *the application of higher moments to loading of parabolic influence lines has great advantages.*

Let the influence line of some quantity T be expressed by a function of abscissa x

$$f(x) = \bar{a}_0 + \bar{a}_1 x + \bar{a}_2\frac{x^2}{2} + \bar{a}_3\frac{x^3}{3!} + \dots \tag{12.1}$$

Then the value of T may be expressed by a function of moments of various orders

$$T = \bar{a}_0 M_0 + \bar{a}_1 M_1 + \bar{a}_2\frac{M_2}{2} + \bar{a}_3\frac{M_3}{3!} + \dots \tag{12.2}$$

Symbols of moments should be replaced in formula (12.2) by quantities corresponding to the given load, according to formula (I) and its particular cases.

It is interesting to compare this method with the usual method of loading influence lines by integration.

Suppose a parabolic influence line of a certain quantity T is loaded over a part of its span by a distributed load (Fig. 12.1).

Let us divide the load into elementary portions $q\,dx$ and calculate the following quantity

$$T=\int\limits_{x=b}^{x=d} q\,dx\cdot f(x)=\int\limits_{x=b}^{x=d}\left(\bar{a}_0+\bar{a}_1x+\bar{a}_2\frac{x^2}{2}+\bar{a}_3\frac{x^3}{3!}+\dots\right)q\,dx \qquad (12.3)$$

For the load presented in Fig. 12.2

$$T=\int\limits_{x=b_1}^{x=d_1}\left(\bar{a}_0+\bar{a}_1x+\bar{a}_2\frac{x^2}{2}+\bar{a}_3\frac{x^3}{3!}\right)q\,dx+$$

$$+\int\limits_{x=b_2}^{x=d_2}\left(\bar{a}_0+\bar{a}_1x+\bar{a}_2\frac{x^2}{2}+\bar{a}_3\frac{x^3}{3!}\right)(q_0+kx)\,dx \qquad (12.4)$$

where q_0 is the ordinate of the triangle diagram at $x=0$.

For determining T by integration and substitution of limit values according to formula (12.4) we have to calculate 24 terms.

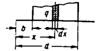


Fig. 12.1

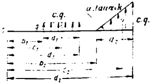


Fig. 12.2

However, if T is calculated without integration by means of higher moments, then formula (12.2) should be used and the final result can be written directly for the case examined

$$T=\bar{a}_0M_0+\bar{a}_1M_1+\bar{a}_2\frac{M_2}{2}+\bar{a}_3\frac{M_3}{3!}=\Big\{\bar{a}_0(qa_1)+$$

$$+\bar{a}_1(qa_1)(-c_1)+\bar{a}_2\left[q\frac{a_1^3}{24}+\frac{(-c_1)^2}{2}(qa_1)\right]+$$

$$+\bar{a}_3\left[(-c_1)\frac{qa_1^3}{24}+\frac{(-c_1)^3}{3!}(qa_1)\right]\Big\}+\Big\{\bar{a}_0\frac{ka_2^2}{2}+\bar{a}_1\frac{ka_2^2}{2}(-c_2)+$$

$$+\bar{a}_2\left[\frac{ka_2^4}{72}+\frac{(-c_2)^2}{2}\cdot\frac{ka_2^2}{2}\right]+$$

$$+\bar{a}_3\left[\frac{ka_2^5}{1620}+(-c_2)\frac{ka_2^4}{72}+\frac{(-c_2)^3}{3!}\cdot\frac{ka_2^2}{2}\right]\Big\} \qquad (12.5)$$

From comparing formulas (12.4) and (12.5), it becomes obvious that integration results in a formula with 24 terms, whereas calculation of T by means of higher moments results in a formula with 13 terms.

Thus, loading parabolic influence lines with algebraic loads by means of higher moments may be carried out without actual integration in solving every engineering problem. The basic formula (I), as well as auxiliary tables and formulas establishing relationships between moments with respect to different axes, make the loading process quite mechanical and allow to write the required result directly in all cases, the number of terms being less than or (seldom) equal to the number of terms when the influence line expression is integrated.

A non-parabolic influence line may be approximated precisely enough by a parabolic influence line with a comparatively small number of terms.

13. FORMULAS FOR STATICALLY INDETERMINATE STRUCTURES

Compiling manuals for designing statically indeterminate beams, frames and arches has always attracted the attention of investigators. However, there was one substantial shortcoming in all these manuals: the same type of frame structure was examined by the authors many times in connection with different loadings; for example, if a frame had n members and m types of loadings, then the author was obliged to give formulas for the reactions and reactive moments (as well as for other necessary parameters) for $m \times n$ different cases. It is obvious, that manuals thus compiled were far from complete, because the number of frames of various types, as well as the number of loadings found in practice, is far too large.

Kleinlogel's manual and other manuals were heavily criticised for containing unnecessary loading cases; for example, loadings shown in Figs. 13.1 and 13.3 are particular cases of the loading shown in Fig. 13.2. The loading of a frame shown in Fig. 13.4 can be derived from Fig. 13.5.

The author of this book suggested a different way of compiling tables, which is based on the application of specially introduced symbols for higher moments. This method makes it possible to examine practically all kinds of loadings acting upon beams and frames in a manual of a substantially reduced size. The principle is as follows: the number of times a frame or a beam of each type is mentioned in the manual should coincide with the number of separate rods in the structure. It will be sufficient then to compile working formulas for successive loading of each rod with a point force; these formulas will be of a general character and will hold for any type of loading acting upon the same rod. This generalization may be carried out by means of symbols introduced for moments of higher order.

Let us examine a T-frame under loadings acting upon its members (Fig. 13.6).

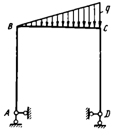


Fig. 13.1

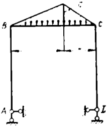


Fig. 13.2

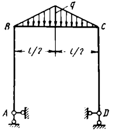


Fig. 13.3

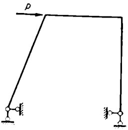


Fig. 13.4

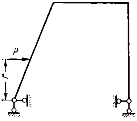


Fig. 13.5

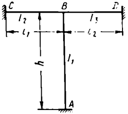


Fig. 13.6.

Loading member CB according to Fig. 13.7

$$M_b^{left} = (1 - i_2k)\frac{Pa^2b}{l_1^2} = \frac{(1 - i_2k)}{l_1^2}(Pa^2l_1 - Pa^3) \tag{13.1}$$

$$M_b^{right} = i_3k\frac{Pa^2b}{l_1^2} = \frac{i_3k}{l_1^2}(Pa^2l_1 - Pa^3) \tag{13.2}$$

$$M_b^{strut} = i_1k\frac{Pa^2b}{l_1^2} = \frac{i_1k}{l_1^2}(Pa^2l_1 - Pa^3) \tag{13.3}$$

$$M_c = (0.5i_2ka + b)\frac{Pab}{l_1^2} = \frac{1}{l_1^2}[(0.5i_2k - 1)a + l_1](l_1 - a)\ Pa =$$

$$= \frac{1}{l_1^2}[(1 - 0.5i_2k)Pa^3 - l_1(2 - 0.5i_2k)Pa^2 + l_1^2Pa] \tag{13.4}$$

$$M_d = i_3k\frac{Pa^2b}{2l_1^2} = \frac{i_3k}{2l_1^2}(Pa^2l_1 - Pa^3) \tag{13.5}$$

$$M_a = i_1k\frac{Pa^2b}{2l_1^2} = \frac{i_1k}{2l_1^2}(Pa^2l_1 - Pa^3) \tag{13.6}$$

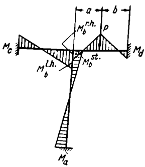


Fig. 13.7

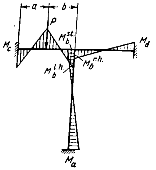


Fig. 13.8

Moments of the first, second, third order of a point load P with respect to point C are equal to

$$M_1 = Pa, \quad M_2 = Pa^2, \quad M_3 = Pa^3$$

By substituting these expressions into formulas (13.1)-(13.6) we obtain

$$M_b^{left} = \frac{1 - i_2k}{l_1^2}(l_1M_2 - M_3) \tag{13.7}$$

$$M_b^{right} = \frac{i_3k}{l_1^2}(l_1M_2 - M_3) \tag{13.8}$$

$$M_b^{strut} = \frac{i_1k}{l_1^2}(l_1M_2 - M_3) \tag{13.9}$$

$$M_c = \frac{1}{l_1^2}\left[(1-0.5i_2k)\,M_3 - l_1\,(2-0.5i_2k)\,M_2 + l_1^2 M_2\right] \quad (13.10)$$

$$M_d = \frac{i_3k}{2l_1^2}\cdot(l_1M_2 - M_3) \quad (13.11)$$

$$M_a = \frac{i_1k}{2l_1^2}\,(l_1M_2 - M_3) \quad (13.12)$$

Loading member BD (Fig. 13.8):

$$M_b^{left} = i_2k\,\frac{Pab^2}{l^2} = \frac{i_2k}{l^2}\,(l_2^2Pa - 2l_2Pa^2 + Pa^3) \quad (13.13)$$

$$M_b^{right} = (1-i_3k)\,\frac{Pab^2}{l_2^2} = \frac{1-i_3k}{l_2^2}\,(l_2^2Pa - 2l_2Pa^2 + Pa^3) \quad (13.14)$$

$$M_b^{strut} = i_2k\,\frac{Pab^2}{l_2^2} = \frac{i_2k}{l_2^2}\,(l_2^2\,Pa - 2l_2Pa^2 + Pa^3) \quad (13.15)$$

$$M_d = (0.5i_3kb + a)\,\frac{Pab}{l_2^2} = \frac{1}{l_2^2}\,[0.5i_3kl_2^2Pa + $$
$$+ l_2\,(1-i_2k)\,Pa^2 - (1-0.5i_2k)\,Pa^3] \quad (13.16)$$

$$M_a = i_1k\,\frac{Pab^2}{2l^2} = \frac{i_1k}{2l^2}\,(l_2^2\,Pa - 2l_2Pa^2 + Pa^3) \quad (13.17)$$

Moments of the first, second and third order of the point load with respect to point B are expressed by

$$M_1 = Pa,\ M_2 = Pa^2,\ M_3 = Pa^3 \quad (13.18)$$

After substituting these expressions into formulas (13.13)-(13.17) we arrive at

$$M_b^{left} = \frac{i_2k}{l_2^2}\,(l_2^2\,M_1 - 2l_2M_2 + M_3) \quad (13.19)$$

$$M_b^{right} = \frac{1-i_3k}{l_2^2}\,(l_2^2M_1 - 2l_2M_2 + M_3) \quad (13.20)$$

$$M_b^{strut} = \frac{i_2k}{l_2^2}\,(l_2^2\,M_1 - 2l_2M_2 + M_3) \quad (13.21)$$

$$M_d = \frac{1}{l_2^2}\,[0.5i_3kl_2^2\,M_1 + l_2\,(1-i_2k)\,M_2 - (1-0.5i_2k)\,M_3] \quad (13.22)$$

$$M_a = \frac{i_1k}{2l^2}\,(l_2^2\,M_1 - 2l_2M_2 + M_3) \quad (13.23)$$

Loading strut AB (Fig. 13.9):

$$M_b^{left} = i_2k\,\frac{Pba^2}{h^2} = \frac{i_2k}{h^2}\,(hPa^3 - Pb^3) \quad (13.24)$$

$$M_b^{right} = i_3k\,\frac{Pba^2}{h^2} = \frac{i_3k}{h^2}\,(hPa^2 - Pa^3) \quad (13.25)$$

$$M_b^{strut} = (1-i_1k)\,\frac{Pba^2}{2h^2} = \frac{1-i_1k}{2h^2}\,(hPa^2 - Pa^3) \quad (13.26)$$

$$M_c = i_2 k \frac{Pba^2}{2h^2} = \frac{i_2 k}{2h^2} (hPa^2 - Pa^3) \tag{13.27}$$

$$M_d = i_3 k \frac{Pba^2}{2h^2} = \frac{i_3 k}{2h^2} (hPa^2 - Pa^3) \tag{13.28}$$

$$M_a = (0.5 i_1 k a + b) \frac{Pab}{h^2} = \frac{1}{h^2} [hPa - \\ - h (2 - 0.5 i_2 k) Pa^2 + (1 - 0.5 i_2 k) Pa^3] \tag{13.29}$$

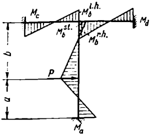


Fig. 13.9

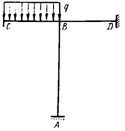


Fig. 13.10

Considering that the expressions for the first, second and third moments of the point load with respect to point A are

$$M_1 = Pa, \ M_2 = Pa^2, \ M_3 = Pa^3$$

we finally obtain the following formulas

$$M_b^{left} = \frac{i_2 k}{h^2} (hM_2 - M_3) \tag{13.30}$$

$$M_b^{right} = \frac{i_3 k}{h^2} (hM_2 - M_3) \tag{13.31}$$

$$M_b^{strut} = \frac{1 - i_1 k}{2h^2} (hM_2 - M_3) \tag{13.32}$$

$$M_c = \frac{i_2 k}{2h^2} (hM_2 - M_3) \tag{13.33}$$

$$M_d = \frac{i_3 k}{2h^2} (hM_2 - M_3) \tag{13.34}$$

$$M_a = \frac{1}{h^2} [h^2 M_1 - h (2 - 0.5 i_2 k) M_2 + (1 - 0.5 i_2 k) M_3] \tag{13.35}$$

In these formulas the following designations are used

$$\left. \begin{array}{c} i_1 = \frac{I_1}{l}; \ i_2 = \frac{I_2}{l_1}; \ i_3 = \frac{I_3}{l_2} \\ k = \frac{1}{i_1 + i_2 + i_3} \end{array} \right\} \tag{13.36}$$

Formulas (13.7)-(13.12), (13.19)-(13.23) and (13.30)-(13.35) are of a general type; they can be used for calculating various loadings acting upon a frame. Consider, for example, a uniformly distributed load applied to member CB (Fig. 13.10). First, the following moments are determined

$$M_1=\frac{ql_1^2}{2},\quad M_2=\frac{ql_1^3}{3},\quad M_3=\frac{ql_1^4}{4} \tag{13.37}$$

After these expressions are substituted into formulas referring to the case of loading CB elements, we arrive at

$$M_b^{left}=\frac{1-i_2k}{l_1^2}\left(l_1\frac{ql_1^3}{3}-\frac{ql_1^4}{4}\right)=(1-i_2k)\frac{ql_1^2}{12} \tag{13.38}$$

$$M_b^{right}=\frac{i_3k}{l_1^2}\left(l_1\frac{ql_1^3}{3}-\frac{ql_1^4}{4}\right)=i_3k\frac{ql_1^2}{12} \tag{13.39}$$

$$M_b^{strut}=\frac{i_1k}{l_1^2}\left(l_1\frac{ql_1^3}{3}-\frac{ql_1^4}{4}\right)=i_1k\frac{ql_1^2}{12} \tag{13.40}$$

$$M_c=(1+i_2k)\frac{ql_1^2}{12} \tag{13.41}$$

$$M_d=i_3k\frac{ql_1^2}{24} \tag{13.42}$$

$$M_a=i_1k\frac{ql_1^2}{24} \tag{13.43}$$

Let us solve the problem illustrated by Fig. 13.11. Moments of the load with respect to point A are expressed as

$$M_1=\frac{qh^2}{6},\quad M_2=\frac{qh^3}{12},\quad M_3=\frac{qh^4}{20} \tag{13.44}$$

which leads to the following set of formulas after these expressions are substituted into the formulas for the loaded strut AB

$$M_b^{left}=\frac{i_2k}{h^2}\left(h\frac{qh^3}{12}-\frac{qh^4}{20}\right)=i_2k\frac{qh^2}{30} \tag{13.45}$$

$$M_b^{right}=\frac{i_3k}{h^2}\left(h\frac{qh^3}{12}-\frac{qh^4}{20}\right)=i_3k\frac{qh^2}{30} \tag{13.46}$$

$$M_b^{strut}=\frac{1-i_1k}{2h^2}\left(h\frac{qh^3}{12}-\frac{qh^4}{20}\right)=(1-i_1k)\frac{qh^2}{60} \tag{13.47}$$

$$M_c=\frac{i_2k}{2h^2}\left(h\frac{qh^3}{12}-\frac{qh^4}{20}\right)=i_2k\frac{qh^2}{60} \tag{13.48}$$

$$M_d=\frac{i_3k}{2h^2}\left(h\frac{qh^3}{12}-\frac{qh^4}{20}\right)=i_3k\frac{qh^2}{60} \tag{13.49}$$

$$M_a=\frac{1}{h^2}\left[h^2\frac{qh^2}{6}-h(2-0.5i_2k)\frac{qh^3}{12}+\right.$$
$$\left.+(1-0.5i_2k)\frac{qh^4}{20}\right]=(3+i_2k)\frac{qh^2}{60} \tag{13.50}$$

Solution for an arbitrary loading of any member may be obtained in a similar manner if the first three moments are determined beforehand.

In the last example only the first three moments were used to calculate the unknown quantities. However, formulas may contain

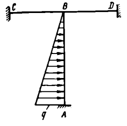


Fig. 13.11

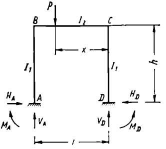


Fig. 13.12

moments of lower and higher order as well. This will be shown in the following problem.

Consider a Π-shaped frame with fixed ends with a load on the strain-beam (Fig. 13.12). The support reaction V_A is given by

$$V_A = \left(a_0 + a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{6}\right) P \qquad (13.51)$$

where

$$a_0 = \frac{6k}{6k+1}, \quad a_1 = \frac{3(2k-1)}{(6k+1)l}, \quad a_2 = \frac{10}{(6k+1)l}$$

$$a_3 = -\frac{12}{(6k+1)l^3}, \quad k = \frac{I_2}{I_1} \; \frac{h}{l}$$

If the following expressions for the moments with respect to point C are taken into account

$$M_0 = P, \ M_1 = Px; \ M_2 = Px^2, \ M_3 = Px^3 \qquad (13.52)$$

then formula (13.51) for the unknown reaction can be written in the general form as

$$V_A = a_0 M_{C0} + a_1 M_{C1} + a_2 \frac{M_{C2}}{2} + a_3 \frac{M_{C3}}{6} \qquad (13.53)$$

Formulas for the other unknown quantities may be generalized accordingly.

Consider an example (Fig. 13.13) in which the unknown quantities are expressed in terms of the first five moments (from 0 to 4).

The axis of the arch is presented by a slightly curved second order parabola. The cross section of the arch follows the law

$$I \cos \alpha = I_c$$

where α is the angle between the tangent to the axis of the arch and the horizontal line, I_c is the moment of inertia of the arch at the key. The unknown quantities are

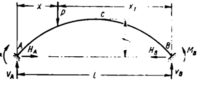


Fig. 13.13

$$A = x_1^2 (l + 2x) \frac{P}{l^3} = (l-x)^2 (l+2x) \frac{P}{l^3} = P - 3\frac{Px^2}{l^2} + 2\frac{Px^3}{l^3} \quad (13.54)$$

$$B = 3\frac{Px^2}{l^2} - 2\frac{Px^3}{l^3} \quad (13.55)$$

$$H = \frac{15}{4} \cdot \frac{k}{f} \left(\frac{Px^2}{l} - 2\frac{Px^3}{l^2} + \frac{Px^4}{l^3} \right) \quad (13.56)$$

$$M_a = -Px + \left(\frac{5}{2}k + 2 \right) \frac{Px^2}{l} - (5k+1) \frac{Px^3}{l^2} + \frac{5k}{2} Px^4 \quad (13.57)$$

$$M_b = \frac{Px^2x_1}{l^3} \left(\frac{5k}{2} x_1 - l \right) = \left(\frac{5k}{2} - 1 \right) \frac{Px^2}{l} -$$

$$- (5k-1) \frac{Px^3}{l^2} + \frac{5k}{2} \cdot \frac{Px^4}{l^3} \quad (13.58)$$

$$M_c = \frac{P}{2l^3} x^2 \left(l^2 - \frac{5k}{2} x_1^2 \right) = \left(1 - \frac{5k}{2} \right) \frac{Px^2}{2l} +$$

$$+ \frac{5k}{2} \cdot \frac{Px^3}{l^2} - \frac{5k}{4} \cdot \frac{Px^4}{l^3} \quad (13.59)$$

In the last formula $0 \leqslant x \leqslant \frac{1}{2} l$.

Coefficient k in these formulas allows for the influence of the longitudinal force and is equal to

$$k = \frac{1}{1+\vartheta}; \quad \vartheta = \frac{45}{4} \cdot \frac{I_c}{F_c f^2} \quad (13.60)$$

Since the expressions for the moments with respect to A are

$$\left. \begin{aligned} & M_{A^0} = P, \quad M_{A^1} = Px, \quad M_{A^2} = Px^2 \\ & M_{A^3} = Px^3, \quad M_{A^4} = Px^4 \end{aligned} \right\} \quad (13.61)$$

we rewrite formulas (13.54)-(13.59) as

$$A = M_{A^0} - \frac{3}{l^2} M_{A^2} + \frac{2}{l^3} M_{A^3} \tag{13.62}$$

$$B = \frac{3}{l^2} M_{A^2} - \frac{2}{l^3} M_{A^3} \tag{13.63}$$

$$H = \frac{15}{4} \cdot \frac{k}{f} \left(\frac{1}{l} M_{A^2} - \frac{2}{l^2} M_{A^3} + \frac{1}{l^3} M_{A^4} \right) \tag{13.64}$$

$$M_a = -M_{A^1} + \left(\frac{5}{2} k + 2 \right) \frac{M_{A^2}}{l} - (5k+1) \frac{M_{A^3}}{l^2} + \frac{5k}{2l^3} M_{A^4} \tag{13.65}$$

$$M_b = \left(\frac{5k}{2} - 1 \right) \frac{M_{A^2}}{l} - (5k-1) \frac{M_{A^3}}{l^2} + \frac{5k}{2} \cdot \frac{M_{A^4}}{l^3} \tag{13.66}$$

$$M_c = \left(1 - \frac{5k}{2} \right) \frac{M_{A^2}}{2l} + \frac{5k}{2} \cdot \frac{M_{A^3}}{l^2} - \frac{5k}{2} \cdot \frac{M_{A^4}}{l^3} \tag{13.67}$$

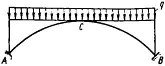


Fig. 13.14

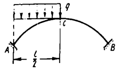


Fig. 13.15

These formulas may be used to determine quantities for the arch shown in Fig. 13.14. In this case

$$\left. \begin{aligned} & M_{A^0} = ql, \quad M_{A^1} = \frac{ql^2}{2}, \quad M_{A^2} = \frac{ql^3}{3} \\ & M_{A^3} = \frac{ql^4}{4}, \quad M_{A^4} = \frac{ql^5}{5} \end{aligned} \right\} \tag{13.68}$$

which leads to the following expression after substituting the values of moments into formulas (13.62)-(13.67)

$$A = ql - \frac{3}{l^2} \; \frac{ql^3}{3} + \frac{2}{l^3} \cdot \frac{ql^4}{4} = \frac{ql}{2} \tag{13.69}$$

In a similar way we obtain

$$B = \frac{ql}{2} \tag{13.70}$$

$$H = \frac{1}{8} \; \frac{kql^3}{f} \tag{13.71}$$

$$M_a = -\frac{k-1}{12} ql^2 \tag{13.72}$$

$$M_b = \left(\frac{5k}{2} - 1 \right) \frac{ql^3}{l \cdot 3} - (5k-1) \frac{ql^4}{l^2 \cdot 4} + \frac{5k}{2l^3} \cdot \frac{ql^5}{5} = -\frac{k-1}{12} ql^2 \tag{13.73}$$

Formula (13.67) cannot be used to determine the bending moment at section C in case of a completely loaded arch, because it is true only for a partially loaded span when $0 \leqslant x \leqslant \frac{l}{2}$. The solution may be obtained by examining the loading diagram of Fig. 13.15. Moments of this loading with respect to section A are expressed as

$$M_{A^2} = \frac{q}{3}\left(\frac{l}{2}\right)^3 = \frac{ql^3}{24} \qquad (13.74)$$

$$M_{A^3} = \frac{q}{4}\left(\frac{l}{2}\right)^4 = \frac{ql^4}{64} \qquad (13.75)$$

$$M_{A^4} = \frac{q}{5}\left(\frac{l}{2}\right)^5 = \frac{ql^5}{160} \qquad (13.76)$$

This particular loading gives the following bending moment M_C^*

$$M_C^* = \frac{1-k}{48} ql^2$$

When the entire arch is loaded the bending moment is twice as large

$$M_C = 2M_C^* = \frac{1-k}{24} ql^2 \qquad (13.77)$$

The method described may be applied to closed contours.

A complete set of formulas for the contour shown in Fig. 13.16 may be compiled by examining only four loading models (Figs. 13.17-13.20). In case of symmetry the number of necessary loading models may be reduced according to the number of axes of symmetry.

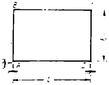


Fig. 13.16

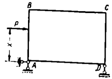


Fig. 13.17

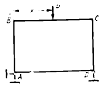


Fig. 13.18

Formulas for the moments at nodes A, B, C, D, as well as for support reactions obtained for the four mentioned loadings, may be used in all design cases.

For example, a distributed load applied to the entire member AB (Fig. 13.21) is taken care of by the formulas composed for the first loading model (Fig. 13.17).

The force acting at the end of a cantilever BB′ may be handled by rejecting the cantilever and by replacing its action with a force and a force couple at node A (Fig. 13.22). Then this force may be considered a particular case of the loading in Fig. 13.17, whereas the force

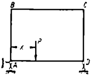


Fig. 13.19

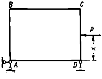


Fig. 13.20

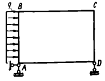


Fig. 13.21

couple at node A may be taken as a particular case of the first or the second loading.

The load acting upon a cantilever BB'' (Fig. 13.23) may be replaced by a force and a force couple acting at node B. The force couple

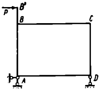


Fig. 13.22

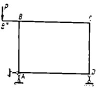


Fig. 13.23

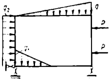


Fig. 13.24

may be taken into account as in the previous case. The force resultant at point B will only cause uniform compression of strut AB.

Finally, the loading shown in Fig. 13.24, as well as any other loading acting upon all four members of the closed contour, may be treat-

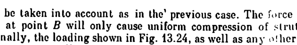


Fig. 13.25

ed as a sum of separate loadings at each member of the closed contour.

For composing formulas for beams and frames with members of variable sections another method can be applied which also makes use of the moment symbols. This method takes into account such factors as resilience of supports, the influence of the transversal force, etc.

Let us apply this method to stepped multispan beams resting on elastic supports. A system of canonical equations of the support moments when the n-th span is under loading (Fig. 13.25) is as follows

$$\left.\begin{aligned}
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&A_{n-3,\,n-5}M_{n-5}+A_{n-3,\,n-4}M_{n-4}+A_{n-3,\,n-3}M_{n-3}+\\
&\qquad+A_{n-3,\,n-2}M_{n-2}+A_{n-3,\,n-1}M_{n-1}=0\\
&A_{n-2,\,n-4}M_{n-4}+A_{n-2,\,n-3}M_{n-3}+A_{n-2,\,n-2}M_{n-2}+\\
&\qquad+A_{n-2,\,n-1}M_{n-1}+A_{n-2,\,n}M_{n}+\overline{\Pi}_{n-2}=0\\
&A_{n-1,\,n-3}M_{n-3}+A_{n-1,\,n-2}M_{n-2}+A_{n-1,\,n-1}M_{n-1}+\\
&\qquad+A_{n-1,\,n}M_{n}+A_{n-1,\,n+1}M_{n+1}+\overline{\Pi}_{n-1}=0\\
&A_{n,\,n-2}M_{n-2}+A_{n,\,n-1}M_{n-1}+A_{n,\,n}M_{n}+\\
&\qquad+A_{n,\,n+1}M_{n+1}+A_{n,\,n+2}M_{n+2}+\overline{\Pi}_{n}=0\\
&A_{n+1,\,n-1}M_{n-1}+A_{n+1,\,n}M_{n}+A_{n+1,\,n+1}M_{n+1}+\\
&\qquad+A_{n+1,\,n+2}M_{n+2}+A_{n+1,\,n+3}M_{n+3}+\overline{\Pi}_{n+1}=0\\
&A_{n+2,\,n}M_{n}+A_{n+2,\,n+1}M_{n+1}+A_{n+2,\,n+2}M_{n+2}+\\
&\qquad+A_{n+2,\,n+3}M_{n+3}+A_{n+2,\,n+4}M_{n+4}=0\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\}\qquad(13.78)$$

In system (13.78) coefficients $A_{m,\,t}$ are expressed by the following formula

$$A_{m,\,t}=6EI_C\delta_{m,\,t}\qquad(13.79)$$

where E = modulus of elasticity of the material of the beam,
I_C = arbitrary number
$\overline{\Pi}_{n-2}, \overline{\Pi}_{n-1}, \overline{\Pi}_{n}, \overline{\Pi}_{n+1}$ = angles between two sections caused by rotation at supports multiplied by $6EI_C$.

The angles are caused not only by deflections of the beams, but also by sagging of the supports. Hence,

$$\left.\begin{aligned}
\overline{\Pi}_{n-1}&=(\Pi_{n-1}+\delta\Pi_{n-1})+\Delta\Pi_{n-1}\\
\overline{\Pi}_{n}&=(\Pi_{n}+\delta\Pi_{n})+\Delta\Pi_{n}
\end{aligned}\right\}\qquad(13.80)$$

where Π_{n-1}, Π_n = angles of rotation at supports $(n-1)$ and n, respectively, due to deflection of the n-th span
$\delta\Pi_{n-1}$, $\delta\Pi_n$ = angles at the same points due to the influence of shearing forces
$\Delta\Pi_{n-1}$, $\Delta\Pi_n$ = angles at the same points due to sagging of supports $(n-1)$ and n

Note. All the angles are multiplied by $6EI_C$.

Sagging of supports $(n-1)$ and n due to the span loading in the basic model is expressed by the following formulas

$$\left.\begin{aligned} y_{n-1} &= R_{n-1}\cdot j_{n-1} \\ y_n &= R_n\cdot j_n \end{aligned}\right\} \tag{13.81}$$

where R_{n-1}, R_n = reactions of supports $(n-1)$ and n
j_{n-1}, j_n = pliancy of supports $(n-1)$ and n

If an angle is considered positive when measured towards the unknown support moment, then the following formulas are obtained

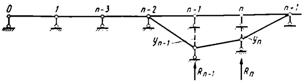


Fig. 13.26

for the angles of rotation of the sections due to sagging of supports (Fig. 13.26)

$$\left.\begin{aligned} \Delta\Pi_{n-2} &= \overline{\Pi}_{n-2} = 6EI_C\,\frac{y_{n-1}}{l_{n-1}} = 6EI_C\,\frac{j_{n-1}}{l_{n-1}}\,R_{n-1} \\ \Delta\Pi_{n-1} &= -6EI_C\,\frac{y_{n-1}}{l_{n-1}} - 6EI_C\,\frac{y_{n-1}}{l_n} + 6EI_C\,\frac{y_n}{l_n} = \\ &= -6EI_C j_{n-1}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)R_{n-1} + 6EI_C\,\frac{j_n}{l_n}\,R_n \\ \Delta\Pi_n &= 6EI_C\,\frac{y_{n-1}}{l_n} - 6EI_C\,\frac{y_n}{l_n} - 6EI_C\,\frac{y_n}{l_{n+1}} = \\ &= 6EI_C\,\frac{j_{n-1}}{l_n}\,R_n - 6EI_C j_n\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right)R_n \\ \Delta\Pi_{n+1} &= \overline{\Pi}_{n+1} = 6EI_C\,\frac{y_n}{l_{n+1}} = 6EI_C\,\frac{j_n}{l_{n+1}}\,R_n \end{aligned}\right\} \tag{13.82}$$

Let us now express R_{n-1} and R_n in terms of higher moments of the span loading.

Suppose M_0, M_1, M_2, M_3 are the zero, first, second and third loading moments with respect to support $(n-1)$ and $\overline{M}_0$, $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$ are moments of the same orders of the span loading with respect to support n. To simplify the symbolics, the following quantities are introduced which reduce the dimensions of all the moments to that of the moment M_1

$$\left.\begin{aligned} \Delta_0 &= l_n M_0; \quad & \Delta_1 &= M_1 \\ \Delta_2 &= \frac{M_2}{l_n}; \quad & \Delta_3 &= \frac{M_3}{l_n^2} \end{aligned}\right\} \tag{13.83}$$

To establish the relations between the loading moments of the span with respect to supports $(n-1)$ and n we derive the following formulas

$$\overline{M}_0 = \int\limits_{(l_n)} q_x\,dx = M_0 = \frac{\Delta_0}{l_n} \tag{13.84}$$

$$\begin{aligned} \overline{M}_1 &= \int\limits_{(l_n)} q_x (l_n - x)\,dx = l_n \int\limits_{(l_n)} q_x d_x - \int\limits_{(l_n)} q_x x\,dx = \\ &= l_n M_0 - M_1 = \Delta_0 - \Delta_1 \end{aligned} \tag{13.85}$$

$$\begin{aligned} \overline{M}_2 &= \int\limits_{(l_n)} q_x (l_n - x)^2\,dx = l_n^2 \int\limits_{(l_n)} q_x\,dx - 2l_n \int\limits_{(l_n)} q_x x\,dx + \\ &+ \int\limits_{(l_n)} x^2 q_x\,dx = l_n^2 M_0 - 2l M_1 + M_2 = l_n (\Delta_0 - 2\Delta_1 + \Delta_2) \end{aligned} \tag{13.86}$$

$$\begin{aligned} \overline{M}_3 &= \int\limits_{(l_n)} q_x (l_n - x)^3\,dx = l_n \int\limits_{(l_n)} q_x\,dx - 3l_n^2 \int\limits_{(l_n)} q_x x\,dx + \\ &+ 3l_n \int\limits_{(l_n)} q_x x^3\,dx - \int\limits_{(l_n)} q_x x^3\,dx = l_n^3 M_0 - 3l_n^2 M_1 + \\ &+ 3l_n M_2 - M_3 = l_n^2 (\Delta_0 - 3\Delta_1 + 3\Delta_2 - \Delta_3) \end{aligned} \tag{13.87}$$

In order to express the reactions of the supports to the span loading in terms of higher moments the first moments of all the forces with respect to supports n and $(n-1)$ are assumed to be zero

$$R_{n-1} l_n - \int\limits_{(l_n)} q_x (l - x)\,dx = 0$$

$$R_n l_n - \int\limits_{(l_n)} q_x x\,dx = 0$$

Hence,

$$R_{n-1} = \frac{1}{l_n} (\Delta_0 - \Delta_1); \quad R_n = \frac{1}{l_n} \Delta_1 \tag{13.88}$$

By introducing designations

$$K_{n-1} = 6EIcj_{n-1}; \quad K_n = 6EIcj_n \tag{13.89}$$

formulas (13.82) become

$$\left.\begin{aligned} \Delta\Pi_{n-2} &= \overline{\Pi}_{n-2} = \frac{K_{n-1}}{l_{n-1}l_n}(\Delta_0-\Delta_1) \\ \Delta\Pi_{n-1} &= -\frac{K_{n-1}}{l_n}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)\Delta_0 + \\ &\quad+\left[\frac{K_n}{l_n^2}+\frac{K_{n-1}}{l_n}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)\right]\Delta_1 \\ \Delta\Pi_n &= \frac{K_{n-1}}{l_n^2}\Delta_0-\left[\frac{K_{n-1}}{l_n^2}+\frac{K_n}{l_n}\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right)\right]\Delta_1 \\ \Delta\Pi_{n+1} &= \overline{\Pi}_{n+1} = \frac{K_n}{l_nl_{n+1}}\Delta_1 \end{aligned}\right\} \qquad (13.82')$$

If the n-th support is fixed, formulas (13.82) may be written as

$$\left.\begin{aligned} \Delta\Pi_{n-2} &= \overline{\Pi}_{n-2} = \frac{K_{n-1}}{l_{n-1}l_n}(\Delta_0-\Delta_1) \\ \Delta\Pi_{n-1} &= -\frac{K_{n-1}}{l_n}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)\Delta_0 + \\ &\quad+\left[\frac{K_n}{l_n^2}+\frac{K_{n-1}}{l_n}\left(\frac{1}{l_{n-1}}+\frac{1}{l_n}\right)\right]\Delta_1 \\ \Delta\Pi_n &= \frac{K_{n-1}}{l_n^2}\Delta_0-\frac{1}{l_n^2}(K_{n-1}+K_n)\Delta_1 \end{aligned}\right\} \qquad (13.90)$$

If support $(n-1)$ is fixed, we have

$$\left.\begin{aligned} \Delta\Pi_{n-1} &= -\frac{K_{n-1}}{l_n^2}\Delta_0+\frac{1}{l_n^2}(K_{n-1}+K_n)\Delta_1 \\ \Delta\Pi_n &= \frac{K_{n-1}}{l_n^2}\Delta_0-\left[\frac{K_{n-1}}{l_n^2}+\frac{K_n}{l_n}\left(\frac{1}{l_n}+\frac{1}{l_{n+1}}\right)\right]\Delta_1 \\ \Delta\Pi_{n+1} &= \overline{\Pi}_{n+1} = \frac{K_n}{l_nl_{n+1}}\Delta_1 \end{aligned}\right\} \qquad (13.91)$$

Finally, when both supports are fixed, which means that a single span beam with fixed ends on elastic supports is considered (Fig. 13.27), formulas (13.82) are reduced to

$$\Delta\Pi_1 = -\Delta\Pi_2 = \frac{K_{n-1}}{l_n^2}\Delta_0+\frac{1}{l_n^2}(K_{n-1}+K_n)\Delta_1 \qquad (13.92)$$

Angles of rotation of sections caused by sagging of elastic supports, as shown by (13.89)-(13.92), depend only on the zero and first moments of the span loading and do not depend on the cross-sectional characteristics of the beams. Therefore these formulas are valid not only for beams of constant sections but for beams of variable sections as well.

In order to determine the angles of rotation at supports $(n-1)$ and n, caused by the loading of the n-th span, moment and transversal force diagrams are drawn for the n-th span loading and unit force couples applied to supports n and $(n-1)$ (Figs. 13.28 and 13.29).

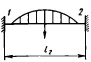


Fig. 13.27

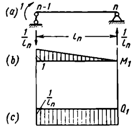


Fig. 13.28

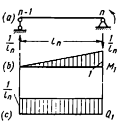


Fig. 13.29

Loaded Span of Constant Cross Section

Angles of rotation are determined by Mohr's integrals

$$\Pi_{n-1}=\frac{6I_C}{I_n}\int_0^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx-\frac{6I_C}{I_n}\int_0^{l_n}\left(1-\frac{x}{l_n}\right)M_p\,dx=$$

$$=-\frac{2I_C}{I_n}\left(1-\frac{x}{l_n}\right)^3\Delta_1 l_n\Big|_0^{l_n}-\frac{6I_C}{I_n}\left(\frac{M_2}{2!}-\frac{M_3}{3!\,l_n}\right)=$$

$$=I_n\frac{I_C}{I_n}(2\Delta_1-3\Delta_2+\Delta_3)\qquad(13.93)$$

$$\Pi_n=\frac{6I_C}{I_n}\int_0^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}dx-\frac{6I_C}{I_n}\int_0^{l_n}\frac{x}{l_n}M_p\,dx=$$

$$=\frac{6I_C}{I_n}\left(\frac{l_n}{2}\Delta_1-\frac{l_n}{3}\Delta_1-\frac{M_3}{3!\,l_n}\right)=l_n\frac{I_C}{I_n}(\Delta_1-\Delta_3)\qquad(13.94)$$

Angles of rotation caused by shearing forces are determined from diagrams on Figs. 13.28-13.30

$$\delta\Pi_{n-1}=-\delta\Pi_n=6I_C\int_0^{l_n}\gamma_n\frac{Q_pQ_1}{I_n}dx=$$

$$=6I_C\int_0^{l_n}\gamma_n\frac{\Delta_1}{l_n^2I_n}dx-6I_C\int_0^{l_n}\gamma_n\frac{Q_p}{l_nI_n}dx=$$

$$= 6I_C\gamma_n\frac{\Delta_1}{l_nI_n} - 6\gamma_n\frac{I_C}{l_nI_n}\left[M_1 - \sum_{i=1}^{m} M_i\right] =$$

$$= 6\,\frac{\gamma_n}{l_n}\cdot\frac{I_C}{I_n}\sum_{i=1}^{n} M_i \tag{13.95}$$

where $\sum_{i=1}^{m} M_i$ is an algebraic sum of external couples applied to span n.

Coefficient γ_n depends on the form and dimensions of the cross

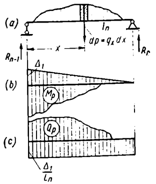


Fig. 13.30

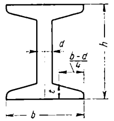


Fig. 13.31

section of span n, as well as on the material of the beam

$$\gamma_n = K\,\frac{E}{G}\cdot\frac{I_n}{F_n}$$

where E, G = moduli of elasticity of the first and second type, respectively

I_n, F_n = moment of inertia and cross-sectional area of span n, respectively

Coefficient K depends on the form of the section according to the following relation

$$K = \frac{F}{I^2}\int_F \frac{S^2(y)}{b(y)}\,dy \tag{13.96}$$

$K = 6/5 = 1.20$ for a rectangular section, $K = 10/9 = 1.11$ for a circular section and $K = 1.03$ for a rhombic section.

For an I-section beam (Fig. 13.31) the following approximate formula is used

$$K=\frac{F}{l^2}\left(S^2\frac{h_1}{d}+\frac{1}{4}Sh_1^3+\frac{1}{120}dh_1^5\right) \qquad (13.97)$$

where

$h_1 = h - 2t - 0.106\,(b + d)$
$S = 0.106b\,(d + 4t)\,(h - 2t - 0.11d + 0.035b)$

For beams of equal spans and constant cross section resting on elastic supports of equal pliancy it is advisable to assume $I_C = \frac{I}{l}$; then the formulas for angles of rotation become

$$\overline{\Pi}_{n-2}=\alpha\,(\Delta_0-\Delta_1)$$

$$\overline{\Pi}_{n-1}=-2\alpha\Delta_0+(2+3\alpha)\,\Delta_1-3\Delta_2+\Delta_3+3\,\frac{\gamma}{l^2}\sum_{i=1}^{m}M_i$$

$$\overline{\Pi}_n=\alpha\Delta_0+(1-3\alpha)\,\Delta_1-\Delta_3-6\,\frac{\gamma}{l^2}\sum_{i=1}^{m}M_i$$

$$\overline{\Pi}_{n+1}=\alpha\Delta_1 \qquad (13.98)$$

where

$$\alpha=6EI\,\frac{1}{l^2}$$

Formulas (13.98) are simplified if the loading of the n-th span is symmetrical about its centre, because

$$\Delta_0 = 2\Delta_1;\quad 2\Delta_1 - 3\Delta_2 + \Delta_3 = \Delta_1 - \Delta_3 \qquad (13.99)$$

Substitution of (13.99) into (13.98) results in

$$\left.\begin{aligned}
\overline{\Pi}_{n-2}&=\alpha\Delta_1\\
\overline{\Pi}_{n-1}&=(1-\alpha)\,\Delta_1-\Delta_3+6\,\frac{\gamma}{l^2}\sum_{i=1}^{m}M_i\\
\overline{\Pi}_n&=(1-\alpha)\,\Delta_1-\Delta_3+6\,\frac{\gamma}{l^2}\sum_{i=1}^{m}M_i\\
\overline{\Pi}_{n+1}&=\alpha\Delta_1
\end{aligned}\right\} \qquad (13.100)$$

Loaded Span of Two-Step Rigidity

(a) *Loading of the first-rigidity segment* (Fig. 13.32). To determine angles of rotation, moment and transversal force diagrams for unit force couples (Figs. 13.28 and 13.29) are applied

$$\Pi_{n-1}=\frac{6I_C}{I_{n1}}\int_0^{a_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx+$$

$$+\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx-\frac{6I_C}{I_{n1}}\int_0^{a_n}\left(1-\frac{x}{l_n}\right)M_p\,dx=$$

$$=l_n\frac{I_C}{I_{n1}}(2\alpha_1-3\Delta_2+\Delta_3) \qquad (13.101)$$

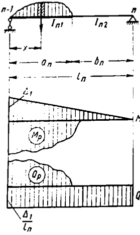


Fig. 13.32

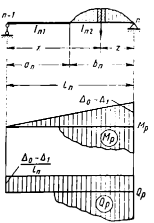


Fig. 13.33

where

$$\alpha_1=1+\frac{b_n^3}{l_n^3}\left(\frac{I_{n1}}{I_{n2}}-1\right)$$

$$\Pi_n=\frac{6I_C}{I_{n1}}\int_0^{a_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}dx+$$

$$+\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}dx-\frac{6I_C}{I_{n1}}\int_0^{a_n}\frac{x}{l_n}M_p\,dx=$$

$$=\frac{I_C}{I_{n1}}(\alpha_2\Delta_1-\Delta_3) \qquad (13.102)$$

where

$$\alpha_2=1+\frac{b_n^2}{l_n^2}\left(\frac{I_{n1}}{I_{n2}}-1\right)\left(3-2\frac{b_n}{l_n}\right)$$

Angle increments caused by shearing forces are determined by the following expressions

$$\delta\Pi_{n-1} = -\delta\Pi_n = \gamma_{n1}\frac{6I_C}{I_{n1}}\int_0^{a_n}\frac{\Delta_1}{l_n^2}\,dx +$$
$$+\gamma_{n2}\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}\frac{\Delta_1}{l_n^2}\,dx - \gamma_{n1}\frac{6I_C}{I_{n1}}\int_0^{a_n}\frac{1}{l_n}Q_p\,dx +$$
$$+\frac{6I_C}{l_n^2}\left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{b_n}{I_{n2}}\right)\sum_{i=1}^{m} M_i =$$
$$= -6\,\frac{I_C}{l_n^2}\left[\gamma_{n1}\frac{l_n}{I_{n1}}\Delta_1 - \left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{b_n}{I_{n2}}\right)\left(\Delta_1 + \sum_{i=1}^{m} M_i\right)\right] \tag{13.103}$$

(b) *Loading of the second-rigidity segment* (Fig. 13.33)

$$\Pi_{n-1} = \frac{6I_C}{I_{n1}}\int_0^{a_n}(\Delta_0 - \Delta_1)\left(1 - \frac{x}{l_n}\right)\frac{x}{l_n}\,dx +$$
$$+\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}(\Delta_0 - \Delta_1)\left(1 - \frac{x}{l_n}\right)\frac{x}{l_n}\,dx - \frac{6I_C}{I_{n2}}\int_0^{b_n}\frac{z}{l_n}M_p\,dz =$$
$$= l_n\frac{I_C}{I_{n2}}\left[1 + \frac{a_n^2}{l_n^2}\left(\frac{I_{n2}}{I_{n1}} - 1\right)\left(3 - 2\frac{a_n}{l_n}\right)\right](\Delta_0 - \Delta_1) -$$
$$-\frac{6I_C}{I_{n2}\cdot l_n}\cdot\frac{\overline{M}_3}{3!}$$

Using formula (13.87) we obtain

$$\Pi_{n-1} = l_n\frac{I_C}{I_{n2}}\left[\alpha_1\Delta_0 + (2 - \alpha_1)\Delta_1 - 3\Delta_2 + \Delta_3\right] \tag{13.104}$$

where

$$\alpha_1 = \frac{a_n^2}{l_n^2}\left(\frac{I_{n2}}{I_{n1}} - 1\right)\left(3 - 2\frac{a_n}{l_n}\right)$$

$$\Pi_n = \frac{6I_C}{I_{n1}}\int_0^{a_n}(\Delta_0 - \Delta_1)\frac{x^2}{l_n^2}\,dx +$$
$$+\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}(\Delta_0 - \Delta_1)\frac{x^2}{l_n^2}\,dx - \frac{6I_C}{I_{n2}}\int_0^{b_n}\left(1 - \frac{z}{l_n}\right)M_p\,dz =$$
$$= \frac{2I_C}{I_{n1}}\cdot\frac{a_n^3}{l_n^2}(\Delta_0 - \Delta_1) + 2l_n\frac{I_C}{I_{n2}}(\Delta_0 - \Delta_1) -$$
$$-\frac{2I_C}{I_{n2}}\cdot\frac{a_n^3}{l_n^2}(\Delta_0 - \Delta_1) - \frac{6I_C}{I_{n2}}\left(\frac{\overline{M}_2}{2!} - \frac{\overline{M}_3}{3!\,l_n}\right)$$

By substituting into this formula expressions for $\overline{M}_2$ and $\overline{M}_3$ from (13.86) and (13.87) we find

$$\begin{aligned}\Pi_n &= 2l_n \frac{I_C}{I_{n2}} \left[1 + \frac{a_n^3}{l_n^3}\left(\frac{I_{n2}}{I_{n1}} - 1\right)\right](\Delta_0 - \Delta_1) - \\ &- l_n \frac{I_C}{I_{n2}} (3\Delta_0 - 6\Delta_1 + 3\Delta_2 - \Delta_0 + 3\Delta_1 - 3\Delta_2 + \Delta_3) = \\ &- l_n \frac{I_C}{I_{n2}} [\alpha_2\Delta_0 + (1 - \alpha_2)\Delta_1 - 3\Delta_2 + \Delta_3] \qquad (13.105)\end{aligned}$$

where

$$\alpha_2 = 2\frac{a_n^3}{l_n^3}\left(\frac{I_{n2}}{I_{n1}} - 1\right)$$

Angle increments caused by shearing forces are determined by the following expressions

$$\begin{aligned}\delta\Pi_{n-1} &= -\delta\Pi_n = -\gamma_{n1}\frac{6I_C}{I_{n1}}\int_0^{l_n}(\Delta_0 - \Delta_1)\frac{1}{l_n^2}\,dx - \\ &- \gamma_{n2}\frac{6I_C}{I_{n2}}\int_{a_n}^{l_n}(\Delta_0 - \Delta_1)\frac{1}{l_n^2}\,dx + \\ &+ \gamma_{n2}\frac{6I_C}{I_{n2}}\int_0^{b_n}\frac{1}{l_n}Q_p\,dz + \\ &+ 6\frac{I_C}{l_n^2}\left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{b_n}{I_{n2}}\right)\sum_{i=1}^{m} M_i = \\ &= -6\frac{I_C}{l_n^2}\left[\gamma_{n2}\frac{l_n}{I_{n2}}(\Delta_0 - \Delta_1) - \right. \\ &\left. - \left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{a_n}{I_{n2}}\right)(\Delta_1 - \Delta_0)\sum_{i=1}^{m} M_i\right] \qquad (13.106)\end{aligned}$$

where

$$\gamma_{n1} = K_{n1}\frac{EI_{n1}}{GF_{n1}}, \quad \gamma_{n2} = K_{n2}\frac{EI_{n2}}{GF_{n2}}$$

and $\sum\limits_{i=1}^{m} M_i$ is an algebraic sum of the given force couples applied to the loaded segment.

Loaded Span of Three-Step Rigidity

(a) *Loading of the first-rigidity segment* (Fig. 13.34). Angle of rotation at support $(n-1)$ is expressed by the following formula

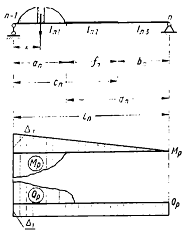


Fig. 13.34

$$\Pi_{n-1}=\frac{6I_C}{I_{n1}}\int\limits_0^{a_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx+\frac{6I_C}{I_{n2}}\int\limits_{a_n}^{c_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx+$$

$$+\frac{6I_C}{I_{n3}}\int\limits_{c_n}^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)^2dx-\frac{6I_C}{I_{n1}}\int\limits_0^{a_n}\left(1-\frac{x}{l_n}\right)M_p\,dx=$$

$$=l_n\frac{I_C}{I_{n1}}(2\alpha_1\Delta_1-3\Delta_2+\Delta_3) \tag{13 107}$$

where

$$\alpha_1=1+\frac{b_n^3}{l_n^3}\left(\frac{I_{n1}}{I_{n3}}-1\right)+\frac{d_n^3}{l_n^3}\left(\frac{I_{n1}}{I_{n2}}-1\right)$$

Angle of rotation at support n is determined as

$$\Pi_n=\frac{6I_C}{I_{n3}}\int\limits_0^{b_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}dx+\frac{6I_C}{I_{n2}}\int\limits_{b_n}^{d_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}dx+$$

$$+\frac{6I_C}{I_{n1}}\int\limits_{d_n}^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}\,dx-\frac{6I_C}{I_{n1}}\int\limits_{0}^{a_n}M_p\frac{x}{l_n}\,dx=$$

$$=l_n\frac{I_C}{I_{n1}}(\alpha_2\Delta_1-\Delta_3) \qquad (13.108)$$

where

$$\alpha_2=1+\frac{b_n^2}{l_n^2}\left(\frac{I_{n1}}{I_{n3}}-\frac{I_{n1}}{I_{n2}}\right)\left(3-2\frac{b_n}{l_n}\right)+$$

$$+\frac{d_n^2}{l_n^2}\left(\frac{I_{n1}}{I_{n2}}-1\right)\left(3-2\frac{d_n}{l_n}\right)$$

Angle increments caused by shearing forces are determined by the following formula

$$\delta\Pi_{n-1}=-\delta\Pi_n=\gamma_{n1}\frac{6I_C}{I_{n1}}\int\limits_{0}^{a_n}\frac{\Delta_1}{l_1^2}\,dx+\gamma_{n2}\frac{6I_C}{I_{n2}}\int\limits_{a_n}^{c_n}\frac{\Delta_1}{l_1^2}\,dx+$$

$$+\gamma_{n3}\frac{6I_C}{I_{n3}}\int\limits_{c_n}^{l_n}\frac{\Delta_1}{l_1^2}\,dx-\gamma_{n1}\frac{6I_C}{I_{n1}}\int\limits_{0}^{c_n}Q_p\frac{l}{l_n}\,dx+$$

$$+\frac{6I_C}{l_n^2}\left(\gamma_{n1}\frac{d_n}{I_{n1}}+\gamma_{n2}\frac{f_n}{I_{n2}}+\gamma_{n3}\frac{b_n}{I_{n3}}\right)\sum_{i=1}^{m}M_i=$$

$$=-6\frac{I_C}{l_n^2}\left[\gamma_{n1}\frac{l_n}{I_{n1}}\Delta_1-\left(\gamma_{n1}\frac{a_n}{I_{n1}}+\gamma_{n2}\frac{f_n}{I_{n2}}+\right.\right.$$

$$\left.\left.+\gamma_{n3}\frac{b_n}{I_{n3}}\right)\left(\Delta_1+\sum_{i=1}^{m}M_i\right)\right] \qquad (13.109)$$

(b) *Loading of the second-rigidity segment* (Fig. 13.35). For calculation conveniences Zhemochkin's and Glushkov's method shall be employed by adding and subtracting the diagram shown in Fig. 13.35*c* to the curvative diagram

$$\Pi_{n-1}=\frac{6I_C}{I_{n3}}\int\limits_{0}^{b_n}\Delta_1\left(\frac{z}{l_n}\right)^2dz+\frac{6I_C}{I_{n2}}\int\limits_{b_n}^{d_n}\Delta_1\left(\frac{z}{l_n}\right)^2dz+$$

$$+\frac{6I_C}{I_{n1}}\int\limits_{d_n}^{l_n}\Delta_1\left(\frac{z}{l_n}\right)^2dz-\frac{6I_C}{I_{n2}}\int\limits_{0}^{c_n}\left(1-\frac{x}{l_n}\right)M_p\,dx+$$

$$+\left[\Delta_1\left(\frac{1}{I_{n2}}-\frac{1}{I_{n1}}\right)\left(2+1-\frac{a_n}{l_n}\right)+\right.$$

$$\left.+\left(\Delta_1-\frac{a_n}{l_n}\Delta_0\right)\left(\frac{1}{I_{n2}}-\frac{1}{I_{n1}}\right)\left(2-2\frac{a_n}{l_n}+1\right)\right]\times$$

$$\times\frac{a_n\cdot 6I_C}{6}=l_n\frac{I_C}{I_{n2}}(\alpha_1\Delta_0+\alpha_2\Delta_1-3\Delta_2+\Delta_3) \qquad (13.110)$$

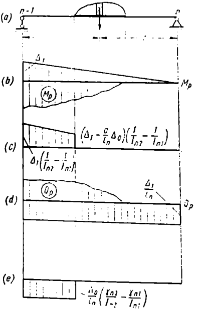


Fig. 13.35

where

$$\alpha_1=\frac{a_n^2}{l_n^2}\left(\frac{I_{n2}}{I_{n1}}-1\right)\left(3-2\frac{a_n}{l_n}\right)$$

$$\alpha_2=2\left[1+\frac{b_n^3}{l_n^3}\left(\frac{I_{n2}}{I_{n1}}-1\right)\right]-\frac{a_n^2}{l_n^2}\left(\frac{I_{n2}}{I_{n1}}-1\right)\left(3-2\frac{a_n}{l_n}\right)$$

$$\Pi_n=\frac{6I_C}{I_{n3}}\int_0^{b_n}\Delta\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}\,dx+\frac{6I_C}{I_{n2}}\int_{b_n}^{d_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}\,dx+$$

$$+\frac{6I_C}{I_{n1}}\int_{d_n}^{l_n}\Delta_1\left(1-\frac{x}{l_n}\right)\frac{x}{l_n}\,dx-\frac{6I_C}{I_{n2}}\int_0^{c_n}\frac{x}{l_n}M_p\,dx+$$

$$+\frac{a_n}{l_n}\left[2\left(\Delta_1-\frac{a_n}{l_n}\Delta_0\right)\left(\frac{1}{I_{n2}}-\frac{1}{I_{n1}}\right)+\right.$$

$$\left.+\Delta_1\left(\frac{1}{I_{n2}}-\frac{1}{I_{n1}}\right)\right]\frac{6I_Ca_n}{6}=$$

$$=l_n\frac{I_C}{I_{n2}}(\alpha_3\Delta_0+\alpha_4\Delta_1-\Delta_3)$$

$$\alpha_3 = 2\frac{a_n^3}{l_n^3}\left(\frac{I_{n2}}{I_{n1}} - 1\right)$$

$$\sigma_4 = 1 - 2\frac{a_n^3}{l_n^3}\left(\frac{I_{n2}}{I_{n1}} - 1\right) + \frac{b_n^2}{l_n^2}\left(\frac{I_{n2}}{I_{n3}} - 1\right)\left(3 - 2\frac{b_n}{l_n}\right)$$

Angle increments caused by shearing forces can be determined by adding the diagram of Fig. 13.35*e* to the diagram of Fig. 13.35*d* and subtracting it from the same diagram

$$\begin{aligned}
\delta\Pi_{n-1} = {} & -\delta\Pi_n = \gamma_{n1}\frac{6I_C}{I_{n1}}\int_0^{a_n}\frac{\Delta_1}{l_n^2}\,dx + \\
& + \gamma_{n2}\frac{6I_C}{I_{n2}}\int_{a_n}^{c_n}\frac{\Delta_1}{l_n^2}\,dx + \gamma_{n3}\frac{6I_C}{I_{n3}}\int_{c_n}^{l_n}\frac{\Delta_1}{l_n^2}\,dx - \\
& - \gamma_{n2}\frac{6I_C}{I_{n2}}\int_0^{c_n} Q_p\,\frac{1}{l_n}\,dx + \frac{6I_C}{l_n^2}\left(\gamma_{n1}\frac{a_n}{I_{n1}} + \right. \\
& \left. + \gamma_{n2}\frac{f_n}{I_{n2}} + \gamma_{n3}\frac{b_n}{I_{n3}}\right)\sum_{i=1}^{m} M_i + \\
& + \frac{\Delta_0}{l_n^2}\left(\frac{\gamma_{n2}}{I_{n2}} - \frac{\gamma_{n1}}{I_{n1}}\right)6I_C a_n = \\
& = -6\frac{I_C}{l_n^2}\Bigl[\gamma_{n2}\frac{l_n}{I_{n2}}\Delta_1 - \left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{f_n}{I_{n2}} + \right. \\
& \left. + \gamma_{n3}\frac{b_n}{I_{n3}}\right)\left(\Delta_1 + \sum_{i=1}^{m} M_i\right) + 6\frac{I_C}{l_n^2}\left(\gamma_{n2}\frac{a_n}{I_{n2}} - \right. \\
& \left. - \gamma_{n1}\frac{a_n}{I_{n1}}\right)\Delta_0
\end{aligned} \qquad (13.111)$$

(c) *Loading of the third-rigidity segment* (Fig. 13.36).

$$\begin{aligned}
\Pi_{n-1} = {} & \frac{6I_C}{I_{n1}}\int_0^{a_n}(\Delta_0 - \Delta_1)\frac{x}{l_n}\left(1 - \frac{x}{l_n}\right)dx + \\
& + \frac{6I_C}{I_{n2}}\int_{a_n}^{c_n}(\Delta_0 - \Delta_1)\frac{x}{l_n}\left(1 - \frac{x}{l_n}\right)dx + \\
& + \frac{6I_C}{I_{n3}}\int_{c_n}^{l_n}(\Delta_0 - \Delta_1)\frac{x}{l_n}\left(1 - \frac{x}{l_n}\right)dx -
\end{aligned}$$

$$-\frac{6I_C}{I_{n3}}\int_0^{b_n} M_p\frac{z}{l_n}\,dz=$$

$$=\frac{6I_C}{l_n}\left(\frac{1}{I_{n1}}-\frac{1}{I_{n2}}\right)\left(\frac{a_n^2}{2}-\frac{a_n^3}{3l_n}\right)(\Delta_0-\Delta_1)+$$

$$+\frac{6I_C}{l_n}\left(\frac{1}{I_{n2}}-\frac{1}{I_{n3}}\right)\left(\frac{c_n^2}{2}-\frac{c_n^3}{3l_n}\right)(\Delta_0-\Delta_1)+$$

$$+l_n\frac{I_C}{I_{n3}}(\Delta_0-\Delta_1)-6\frac{I_C}{I_{n3}}\frac{\bar{M}_3}{3l\,l_n}$$

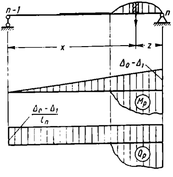


Fig. 13.36

After the expression for $\bar{M}_3$ is substituted into this formula the latter may be finally written as

$$\Pi_{n-1}=l_n\frac{I_C}{I_{n3}}\left[\alpha_1\Delta_0+(2-\alpha_1)\Delta_1-3\Delta_2+\Delta_3\right] \qquad (13.112)$$

where

$$\alpha_1=\frac{a_n^2}{l_n^2}\left(\frac{I_{n3}}{I_{n1}}-\frac{I_{n3}}{I_{n2}}\right)\left(3-2\frac{a_n}{l_n}\right)+\frac{c_n^2}{l_n^2}\left(\frac{I_{n3}}{I_{n2}}-1\right)\left(3-2\frac{c_n}{l_n}\right)$$

$$\Pi_n=\frac{6I_C}{I_{n1}}\int_0^{a_n}(\Delta_0-\Delta_1)\frac{x^2}{l_n^2}\,dx+\frac{6I_C}{I_{n2}}\int_{a_n}^{c_n}(\Delta_0-\Delta_1)\frac{x^2}{l_n^2}\,dx+$$

$$+\frac{6I_C}{I_{n3}}\int_{c_n}^{l_n}(\Delta_0-\Delta_1)\frac{x^2}{l_n^2}\,dx-\frac{6I_C}{I_{n3}}\int_0^{b_n}M_p\left(1-\frac{z}{l_n}\right)dz=$$

$$=l_n\frac{I_C}{I_{n3}}\left[\alpha_2\Delta_0+(1-\alpha_2)\Delta_1-\Delta_3\right] \qquad (13.113)$$

where

$$\alpha_2 = 2\frac{a_n^3}{l_n^3}\left(\frac{I_{n3}}{I_{n1}} - \frac{I_{n3}}{I_{n2}}\right) + 2\frac{c_n^3}{l_n^3}\left(\frac{I_{n3}}{I_{n2}} - 1\right)$$

Angles of rotation caused by shearing forces are

$$\begin{aligned}\delta\Pi_{n-1} = -\delta\Pi_n = &-\gamma_{n1}\frac{6I_C}{I_{n1}}\int_0^{a_n}\frac{\Delta_0-\Delta_1}{l_n^2}\,dx - \gamma_{n2}\frac{6I_C}{I_{n2}}\int_{a_n}^{c_n}\frac{\Delta_0-\Delta_1}{l_n^2}\,dx - \\ &-\gamma_{n3}\frac{6I_C}{I_{n3}}\int_{c_n}^{l_n}\frac{\Delta_0-\Delta_1}{l_n^2}\,dx + \gamma_{n3}\frac{6I_C}{I_{n3}}\int_0^{b_n}Q_p\frac{1}{l_n}\,dz + \\ &+\frac{6I_C}{l_n^2}\left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{f_n}{I_{n2}} + \gamma_{n3}\frac{b_n}{I_{n3}}\right)\sum_{i=1}^{m} M_i = \\ &= -\frac{6I_C}{l_n^2}\left[\gamma_{n3}\frac{l_n}{I_{n3}}(\Delta_0-\Delta_1) - \left(\gamma_{n1}\frac{a_n}{I_{n1}} + \gamma_{n2}\frac{f_n}{I_{n2}} + \right.\right. \\ &\left.\left. + \gamma_{n3}\frac{b_n}{I_{n3}}\right)\left(\Delta_1 - \Delta_0 + \sum_{i=1}^{m} M_i\right)\right] \qquad (13.114)\end{aligned}$$

In these formulas $\sum_{i=1}^{m} M_i$ is an algebraic sum of external bending moments applied to the loaded segment. Moments are assumed

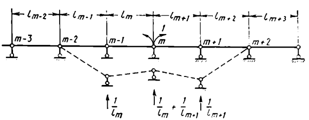


Fig. 13.37

positive when directed clockwise and negative when directed counterclockwise.

Coefficients of system (13.78) are angles of rotation of a section over support m due to a unit force couple applied at support t (multiplied by $6EI_C$). It may be easily proved that $A_{m,\,t} = A_{t,\,m}$. That is why the matrix of system (13.78) is symmetrical about the main diagonal.

Let us determine these coefficients as a sum of three terms (Fig. 13.37)

$$A_{m,\,t} = A^1_{m,\,t} + A^2_{m,\,t} + A^3_{m,\,t} \tag{13.115}$$

where $A^1_{m,\,t}$ = angle of rotation due to flexure
$A^2_{m,\,t}$ = angle of rotation due to shear
$A^3_{m,\,t}$ = angle of rotation due to sagging of supports

$$\ldots A_{m,\,m-4} = A_{m,\,m-3} = 0$$

$$A_{m,\,m-2} = A^3_{m,\,m-2} = 6EI_C\,\frac{j_{m-1}}{l_m l_{m-1}} = \frac{K_{m-1}}{l_m l_{m-1}} \tag{13.116}$$

$$A_{m,\,m-1} = A^1_{m,\,m-1} + A^2_{m,\,m-1} + A^3_{m,\,m-1} \tag{13.117}$$

$$A_{m,\,m} = A^1_{m,\,m} + A^2_{m,\,m} + A^3_{m,\,m} \tag{13.118}$$

$$A_{m,\,m+1} = A^1_{m,\,m+1} + A^2_{m,\,m+1} + A^3_{m,\,m+1} \tag{13.119}$$

$$A_{m,\,m+2} = A^3_{m,\,m+2} = 6EI_C\,\frac{j_{m+2}}{l_{m+1} l_{m+2}} = \frac{K_{m+2}}{l_{m+1} l_{m+2}} \tag{13.120}$$

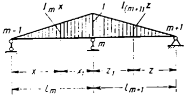


Fig. 13.38

Coefficients $A^1_{m,\,m-1}$, $A^1_{m,\,m}$, and $A^1_{m,\,m+1}$ are determined according to Fig. 13.38 by the following formulas

$$A^1_{m,\,m-1} = 6I_C \int\limits_0^{l_m} \frac{\frac{x}{l_m}\left(1-\frac{x}{l_m}\right)dx}{I_{mx}} = \frac{6I_C}{l_m}\int\limits_0^{l_m}\frac{1}{I_{mx}}\,x\,dx -$$

$$-\frac{6I_C}{l_m^2}\int\limits_0^{l_m}\frac{1}{I_{mx}}\,x^2\,dx = \frac{6I_C}{l_m}\,M^{(m)}_{(m-1)1} - \frac{6I_C}{l_m^2}\,M^{(m)}_{(m-1)2} \tag{13.121}$$

$$A^1_{m,\,m} = 6I_C\int\limits_0^{l_m}\frac{\left(\frac{x}{l_m}\right)^2dx}{I_{mx}} + 6I_C\int\limits_0^{l_{m+1}}\frac{\left(\frac{z}{l_{m+1}}\right)^2dz}{I_{(m+1)z}} =$$

$$= \frac{6I_C}{l_m^2}\int\limits_0^{l_m}\frac{1}{I_{mx}}\,x^2\,dx + \frac{6I_C}{l_{m+1}^2}\int\limits_0^{l_{m+1}}\frac{z^2\,dz}{I_{(m+1)z}} =$$

$$= \frac{6I_C}{l_m^2}\,M^{(m)}_{(m-1)2} + \frac{6I_C}{l_{m+1}^2}\,M^{(m+1)}_{(m+1)2} \tag{13.122}$$

$$A^1_{m,\,m+1} = 6I_C \int\limits_0^{l_{m+1}} \frac{\left(1-\frac{z}{l_{m+1}}\right)\frac{z}{l_{m+1}}\,dz}{I_{(m+1)^z}} =$$

$$= \frac{6I_C}{l_{m+1}} \int\limits_0^{l_{m+1}} \frac{1}{I_{(m+1)^z}}\, z\,dz - \frac{6I_C}{l^2_{m+1}} \int\limits_0^{l_{m+1}} \frac{1}{I_{(m+1)^z}}\, z^2\,dz =$$

$$= \frac{6I_C}{l_{m+1}} M^{(m+1)}_{(m+1)1} - \frac{6I_C}{l^2_{m+1}} M^{(m+1)}_{(m+1)2} \qquad (13.123)$$

Formulas (13.121)-(13.123) can be obtained in a different form

$$A^1_{m,\,m-1} = 6I_C \int\limits_0^{l_m} \frac{\frac{x_1}{l_m}\left(1-\frac{x_1}{l_m}\right)dx_1}{I_{mx_1}} = \frac{6I_C}{l_m} M^{(m)}_{m1} - \frac{6I_C}{l^2_m} M^{(m)}_{m2} \qquad (13.121')$$

$$A^1_{m,\,m} = 6I_C \int\limits_0^{l_m} \frac{\left(1-\frac{x_1}{l_m}\right)^2 dx_1}{I_{mx_1}} + 6I_C \int\limits_0^{l_{m+1}} \frac{\left(1-\frac{z_1}{l_{m+1}}\right)^2 dz_1}{I_{(m+1)z_1}} =$$

$$= 6I_C M^{(m)}_{m0} - \frac{12I_C}{l_m} M^{(m)}_{m1} + \frac{6I_C}{l^2_m} M^{(m)}_{m2} +$$

$$+ 6I_C M^{(m+1)}_{m0} - \frac{12I_C}{l_{m+1}} M^{(m+1)}_{m1} + \frac{6I_C}{l^2_{m+1}} M^{(m+1)}_{m2} \qquad (13.122')$$

$$A^1_{m,\,m+1} = 6I_C \int\limits_0^{l_{m+1}} \frac{\left(1-\frac{z_1}{l_{m+1}}\right)\frac{z_1}{l_{m+1}}\,dz_1}{I_{(m+1)z_1}} =$$

$$= \frac{6I_C}{l_{m+1}} M^{(m+1)}_{m1} - \frac{6I_C}{l^2_{m+1}} M^{(m+1)}_{m2} \qquad (13.123')$$

Here

$M^{(m)}_{(m-1)1}, M^{(m)}_{(m-1)2}$ = first and second moments of flexibility diagram of span m with respect to support $m-1$ (Fig. 13.39)

$M^{(m)}_{m0},\ M^{(m)}_{m1},\ M^{(m)}_{m2}$ = zero, first and second moments of flexibility diagram of span m with respect to support m

$M^{(m+1)}_{m1},\ M^{(m+1)}_{m2}$ = first and second moments of flexibility diagram of span $(m+1)$ with respect to support m

$M^{(m+1)}_{(m+1)1},\ M^{(m+1)}_{(m+1)2}$ = first and second moments of flexibility diagram of span $(m+1)$ with respect to support $m+1$

Coefficients $A^2_{m,\,m-1}$, $A^2_{m,\,m}$, $A^2_{m,\,m+1}$ are determined by the following expressions

$$A^2_{m,\,m-1} = -6I_C \int\limits_0^{l_m} \gamma_m x_1 \frac{\frac{1}{l_m}\cdot\frac{1}{l_m}\,dx_1}{I_{mx_1}} =$$

$$= -\frac{6I_C}{l_m^2} \int\limits_0^{l_m} \frac{\gamma_{mx_1}}{I_{mx_1}}\,dx_1 = -\frac{6I_C}{l_m^2}\,\overline{M}^m_{m0} \qquad (13.124)$$

$$A^2_{m,\,m} = \frac{6I_C}{l_m^2}\,\overline{M}^{(m)}_{m0} + \frac{6I_C}{l_{m+1}^2}\,\overline{M}^{(m+1)}_{m0} \qquad (13.125)$$

$$A^2_{m,\,m+1} = -6I_C \int\limits_0^{l_{m+1}} \gamma_{(m+1)}\, z_1 \frac{\frac{1}{l_{m+1}}\cdot\frac{1}{l_{m+1}}}{I_{mz_1}}\,dz_1 =$$

$$= -\frac{6I_C}{l_{m+1}^2}\,\overline{M}^{(m+1)}_{m0} \qquad (13.126)$$

Here $\overline{M}^{(m)}_{m0}$, $\overline{M}^{(m+1)}_{m0}$ are moments of zero order (area) of the $\frac{\gamma}{I}$

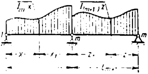
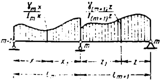


Fig. 13.39



$\frac{\gamma_m x}{I_m x}$
$\frac{\gamma_{(m+1)} z}{I_{(m+1)} z}$
m-1
m
m+1
x
x_1
z_1
z
l_m
l_{m+1}

Fig. 13.40

diagram (Fig. 13.40) of the m-th and $(m+1)$-th spans with respect to the m-th support.

The following expressions determine coefficients $A^3_{m,\,m-1}$, $A^3_{m,\,m}$ and $A^3_{m,\,m+1}$ which depend on the pliability of supports (Fig. 13.41)

$$A^3_{m,\,m-2} = A^3_{m-2,\,m} = 6EI_C\,\frac{j_{m-1}}{l_m l_{m-1}} = \frac{K_{m-1}}{l_m l_{m-1}} \qquad (13.127)$$

$$A^3_{m,\,m-1} = A^3_{m-1,\,m} = 6EI_C\left[-\frac{j_{m-1}}{l_m}\left(\frac{1}{l_{m-1}}+\frac{1}{l_m}\right)-\right.$$

$$\left.-\frac{j_m}{l_m}\left(\frac{1}{l_m}+\frac{1}{l_{m+1}}\right)\right] = -\frac{K_{m-1}}{l_m}\left(\frac{1}{l_{m-1}}+\frac{1}{l_m}\right)-$$

$$-\frac{K_m}{l_m}\left(\frac{1}{l_m}+\frac{1}{l_{m+1}}\right) \qquad (13.128)$$

$$A^3_{m,\,m} = 6EI_C\left[\frac{j_{m-1}}{l_m l_m} + \frac{j_{m+1}}{l_{m+1}l_{m+1}} + j_m\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right)\frac{1}{l_m} + \right.$$

$$\left. + j_m\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right)\frac{1}{l_{m+1}}\right] =$$

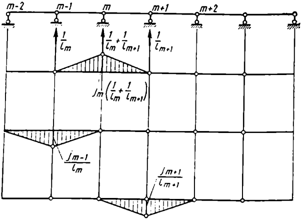
$$= \frac{K_{m-1}}{l^2_m} + K_m\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right)^2 + \frac{K_{m+1}}{l^2_{m+1}} \qquad (13.129)$$

$$A^3_{m,\,m+1} = A^3_{m+1,\,m} = 6EI_C\left[-\frac{j_m}{l_{m+1}}\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right) - \right.$$

$$\left. - \frac{j_{m+1}}{l_{m+1}}\left(\frac{1}{l_{m+1}} + \frac{1}{l_{m+2}}\right)\right] =$$

$$= -\frac{K_m}{l_{m+1}}\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right) - \frac{K_{m+1}}{l_{m+1}}\left(\frac{1}{l_{m+1}} + \frac{1}{l_{m+2}}\right) \qquad (13.130)$$

$$A^3_{m,\,m+2} = 6EI_C\,\frac{j_{m+1}}{l_{m+1}l_{m+2}} = \frac{K_{m+1}}{l_{m+1}l_{m+2}} \qquad (13.131)$$



m-2
m-1
m
m+1
m+2
$\frac{1}{l_m}$
$\frac{1}{l_m}+\frac{1}{l_{m+1}}$
$\frac{1}{l_{m+1}}$
$j_m\left(\frac{1}{l_m}+\frac{1}{l_{m+1}}\right)$
$\frac{j_{m-1}}{l_m}$
$\frac{j_{m+1}}{l_{m+1}}$

Fig. 13.41

If the $(m-1)$-th support is fixed, then coefficients $A^3_{m,\,m-2}$ and $A^3_{m,\,m-1}$ become

$$\left.\begin{aligned} A^3_{m,\,m-2} &= 0 \\ A^3_{m,\,m-1} &= -\frac{K_{m-1}}{l^2_m} - \frac{K_m}{l_m}\left(\frac{1}{l_m} + \frac{1}{l_{m+1}}\right) \end{aligned}\right\} \qquad (13.1\,28')$$

Coefficients $A^3_{m,\,m}$, $A^3_{m,\,m+1}$ and $A^3_{m,\,m+2}$ do not change.

If support m is fixed (in this case $m = 1$), then

$$A^3_{1,\,1} = \frac{K_1}{l_2^2} + \frac{K_2}{l_2^2} \tag{13.129'}$$

$$A^3_{1,\,2} = -\frac{K_1}{l_2^2} - \frac{K_2}{l_2^2}\left(\frac{1}{l_2} + \frac{1}{l_3}\right) \tag{13.130'}$$

$$A^3_{1,\,3} = \frac{K_2}{l_2 l_3} \tag{13.131'}$$

If support $(m + 1)$ is fixed the coefficients change in a similar way.

Coefficients $A^1_{m.\,p}$ and $A^2_{m.\,p}$ will be calculated now for some types of beams. First, formulas will be derived for higher moments with

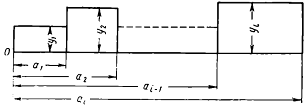


Fig. 13.42

respect to an axis passing through point O of a figure composed of rectangles (Fig. 13.42)

$$M_{O0} = y_1 a_1 + y_2 (a_2 - a_1) + \ldots + y_i (a_i - a_{i-1}) = \\ = \sum_{h=1}^{i} \frac{y_k (a_k - a_{k-1})}{1} \tag{13.132}$$

$$M_{O1} = y_1 \frac{a_1^2}{2} + y_2 \frac{a_2^2 - a_1^2}{2} + \ldots + y_i \frac{a_i^2 - a_{i-1}^2}{2} = \\ = \sum_{h=1}^{i} \frac{y_k (a_k^2 - a_{k-1}^2)}{2} \tag{13.133}$$

. . . .

$$M_{O^s} = \sum_{h=1}^{i} \frac{y_k (a_k^{s-1} - a_{k-1}^{s-1})}{2} \tag{13.134}$$

Examine a beam of constant cross section within each span (Fig. 13.43). The higher moments in the calculation formulas are determined by the following expressions

$$M^{(m)}_{(m-1)1} = \frac{1}{I_m}\cdot\frac{l_m^2}{2}, \quad M^{(m)}_{(m-1)2} = \frac{1}{I_m}\cdot\frac{l_m^3}{3}$$

$$M^{(m+1)}_{(m+1)1} = \frac{1}{I_{m+1}}\cdot\frac{l_{m+1}^2}{2}, \quad M^{(m+1)}_{(m+1)2} = \frac{1}{I_{m+1}}\cdot\frac{l_{m+1}^3}{3}$$

$$\overline{M}^{(m)}_{m0} = l_m\frac{\gamma_m}{I_m}, \quad \overline{M}^{(m+1)}_{m0} = l_{m+1}\frac{\gamma_{m+1}}{I_{m+1}}$$

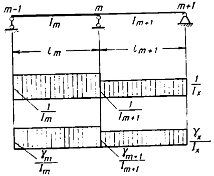


Fig. 13.43

Having substituted these expressions into (13.121)-(13.123), we obtain

$$A^1_{m,\,m-1} = \frac{6I_C}{l_m}\cdot\frac{1}{I_m}\cdot\frac{l_m^2}{2} - \frac{6I_C}{l_m^2}\cdot\frac{1}{I_m}\cdot\frac{l_m^3}{3} = l_m\frac{I_C}{I_m} \tag{13.135}$$

$$A^1_{m,\,m} = 2\left(l_m\frac{I_C}{I_m} + l_{m+1}\frac{I_C}{I_{m+1}}\right) \tag{13.136}$$

$$A^1_{m,\,m+1} = \frac{6I_C}{l_{m+1}}\cdot\frac{1}{I_{m+1}}\cdot\frac{l_{m+1}^2}{2} - \frac{6I_C}{l_{m+1}^2}\cdot\frac{1}{I_{m+1}}\cdot\frac{l_{m+1}^3}{3} =$$

$$= l_{m+1}\cdot\frac{I_C}{I_{m+1}} \tag{13.137}$$

$$A^2_{m,\,m-1} = -\frac{6I_C}{l_m^2}\,l_m\frac{\gamma_m}{I_m} = -6\,\frac{\gamma_m}{l_m}\cdot\frac{I_C}{I_m} \tag{13.138}$$

$$A^2_{m,\,m} = 6\left(\frac{\gamma_m}{l_m}\cdot\frac{I_C}{I_m} + \frac{\gamma_{m+1}}{l_{m+1}}\cdot\frac{I_C}{I_{m+1}}\right) \tag{13.139}$$

$$A^2_{m,\,m+1} = -\frac{6I_C}{l_{m+1}^2}\,l_{m+1}\frac{\gamma_{m+1}}{I_{m+1}} = -6\,\frac{\gamma_{m+1}}{l_{m+1}}\;\frac{I_C}{I_{m+1}} \tag{13.140}$$

Examine a beam with two different cross sections within one span (Fig. 13.44). The higher moments in formulas (13.121)-(13.123) are determined by the following expressions

$$M^{(m)}_{(m-1)1} = \frac{1}{I_{m,1}} \cdot \frac{a_m^2}{2} + \frac{1}{I_{m,2}} \cdot \frac{l_m^2 - a_m^2}{2}$$

$$M^{(m)}_{(m-1)2} = \frac{1}{I_{m,1}} \cdot \frac{a_m^3}{3} + \frac{1}{I_{m,2}} \cdot \frac{l_m^3 - a_m^3}{3}$$

$$M^{(m+1)}_{m1} = \frac{1}{I_{(m+1),1}} \cdot \frac{a_{m+1}^2}{2} + \frac{1}{I_{(m+1),2}} \cdot \frac{l_{m+1}^2 - a_{m+1}^2}{2}$$

$$M^{(m+1)}_{m2} = \frac{1}{I_{(m+1),1}} \cdot \frac{a_{m+1}^3}{3} + \frac{1}{I_{(m+1),2}} \cdot \frac{l_{m+1}^3 - a_{m+1}^3}{3}$$

$$M^{(m+1)}_{(m+1)1} = \frac{1}{I_{(m+1),2}} \cdot \frac{b_{m+1}^2}{2} + \frac{1}{I_{(m+1),1}} \cdot \frac{l_{m+1}^2 - b_{m+1}^2}{2}$$

$$M^{(m+1)}_{(m+1)2} = \frac{1}{I_{(m+1),2}} \cdot \frac{b_{m+1}^3}{3} + \frac{1}{I_{(m+1),1}} \cdot \frac{l_{m+1}^3 + b_{m+1}^3}{3}$$

$$\overline{M}^{(m)}_{m0} = a_m \frac{\gamma_{m,1}}{I_{m,1}} + b_m \frac{\gamma_{m,2}}{I_{m,2}}$$

$$\overline{M}^{(m+1)}_{m0} = a_{m+1} \frac{\gamma_{(m+1),1}}{I_{(m+1),1}} + b_{m+1} \frac{\gamma_{(m+1),2}}{I_{(m+1),2}}$$

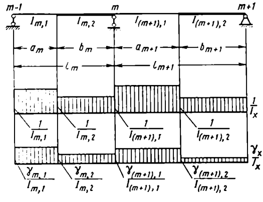


Fig. 13.44

Substituting these values into formulas (13.121)-(13.123) we obtain

$$A^1_{m,m-1} = \frac{3I_C}{l_m}\left(\frac{a_m^2}{I_{m,1}} + \frac{l_m^2 - a_m^2}{I_{m,2}}\right) - \frac{2I_C}{l_m^2}\left(\frac{a_m^3}{I_{m,1}} + \frac{l_m^3 - a_m^3}{I_{m,2}}\right) =$$

$$\begin{aligned}
&= 3l_m \frac{I_C}{I_{m,2}} \left[1 + \frac{a_m^2}{l_m^2} \left(\frac{I_{m,2}}{I_{m,1}} - 1 \right) \right] - \\
&- 2l_m \frac{I_C}{I_{m,2}} \left[1 + \frac{a_m^3}{l_m^3} \left(\frac{I_{m,2}}{I_{m,1}} - 1 \right) \right] = \\
&= l_m \frac{I_C}{I_{m,2}} \left[1 + \frac{a_m^2}{l_m^2} \left(\frac{I_{m,2}}{I_{m,1}} - 1 \right) \left(3 - 2\frac{a_m}{l_m} \right) \right] \qquad (13.141)
\end{aligned}$$

$$\begin{aligned}
A_{m,m}^1 &= 2\frac{I_C}{l_m^2} \left(\frac{a_m^3}{I_{m,1}} + \frac{l_m^3 - a_m^3}{I_{m,2}} \right) + \\
&+ 2\frac{I_C}{l_{m+1}^2} \left(\frac{b_{m+1}^3}{I_{(m+1),2}} + \frac{l_{m+1}^3 - b_{m+1}^3}{I_{(m+1),1}} \right) = \\
&= 2l_m \frac{I_C}{I_{m,2}} \left[1 + \frac{a_m^3}{l_m^3} \left(\frac{I_{m,2}}{I_{m,1}} - 1 \right) \right] + \\
&+ 2l_{m+1} \frac{I_C}{I_{(m+1),1}} \left[1 + \frac{b_{m+1}}{l_m^3} \left(\frac{I_{(m+1),1}}{I_{(m+1),2}} \right) \right] \qquad (13.142)
\end{aligned}$$

$$\begin{aligned}
A_{m,m+1}^2 &= \frac{3I_C}{l_{m+1}} \left(\frac{b_{m+1}^2}{I_{(m+1),2}} + \frac{l_{m+1}^2 - b_{m+1}^2}{I_{(m+1),1}} \right) - \\
&- \frac{2I_C}{l_{m+1}^2} \left(\frac{b_{m+1}^3}{I_{(m+1),2}} + \frac{l_{m+1}^3 - b_{m+1}^3}{I_{(m+1),1}} \right) = \\
&= 3l_{m+1} \frac{I_C}{I_{(m+1),1}} \left[1 + \frac{b_{m+1}^2}{l_{m+1}^2} \left(\frac{I_{(m+1),1}}{I_{(m+1),2}} - 1 \right) \right] - \\
&- 2l_{m+1} \frac{I_C}{I_{(m+1),1}} \left[1 + \frac{b_{m+1}^3}{l_{m+1}^3} \left(\frac{I_{(m+1),1}}{I_{(m+1),2}} - 1 \right) \right] = \\
&= l_{m+1} \frac{I_C}{I_{(m+1),1}} \left[1 + \frac{b_{m+1}^2}{l_{m+1}^2} \left(\frac{I_{(m+1),1}}{I_{(m+1),2}} - 1 \right) \left(3 - 2\frac{b_{m+1}}{l_{m+1}} \right) \right] \\
&\qquad (13.143)
\end{aligned}$$

$$A_{m,m-1}^2 = -\frac{6I_C}{l_m^2} \left(a_m \frac{\gamma_{m,1}}{I_{m,1}} + b_m \frac{\gamma_{m,2}}{I_{m,2}} \right) \qquad (13.144)$$

$$\begin{aligned}
A_{m,m}^2 &= \frac{6I_C}{l_m^2} \left(a_m \frac{\gamma_{m,1}}{I_{m,1}} + b_m \frac{\gamma_{m,2}}{I_{m,2}} \right) + \\
&+ \frac{6I_C}{l_{m+1}^2} \left(a_{m+1} \frac{\gamma_{(m+1),1}}{I_{(m+1),1}} + b_{m+1} \frac{\gamma_{(m+1),2}}{I_{(m+1),2}} \right) \qquad (13.145)
\end{aligned}$$

$$A_{m,m+1}^2 = -\frac{6I_C}{l_{m+1}^2} \left(a_{m+1} \frac{\gamma_{(m+1),1}}{I_{(m+1),1}} + b_{m+1} \frac{\gamma_{(m+1),2}}{I_{(m+1),2}} \right) \qquad (13.146)$$

Similar formulas may be obtained for beams having s different cross sections within each span (Fig. 13.45). Leaving out the intermediate calculations, we write the final formulas for $s = 3$

$$\begin{aligned}
A_{m,m-1}^1 &= l_m \frac{I_C}{I_{m,3}} \left[1 + \frac{a_m^2}{l_m^2} \left(\frac{I_{m,3}}{I_{m,1}} - \frac{I_{m,3}}{I_{m,2}} \right) \left(3 - 2\frac{a_m}{l_m} \right) + \right. \\
&\left. + \frac{c_m^2}{l_m^2} \left(\frac{I_{m,3}}{I_{m,2}} - 1 \right) \left(3 - 2\frac{c_m}{l_m} \right) \right] \qquad (13.147)
\end{aligned}$$

$$\begin{aligned}A^1_{m,m} = {} & 2l_m \frac{I_C}{I_{m,3}} \Big[1 + \frac{a_m^3}{l_m^3} \left(\frac{I_{m,3}}{I_{m,1}} - \frac{I_{m,3}}{I_{m,2}} \right) + \\ & + \frac{c_m^3}{l_m^3} \left(\frac{I_{m,3}}{I_{m,2}} - 1 \right) \Big] + \\ & + 2l_{m+1} \frac{I_C}{I_{(m+1),1}} \Big[1 + \frac{b_{m+1}^3}{l_{m+1}^3} \left(\frac{I_{m+1,1}}{I_{m+1,3}} - \frac{I_{m+1,1}}{I_{m+1,2}} \right) + \\ & + \frac{d_{m+1}^3}{l_{m+1}^3} \left(\frac{I_{m+1,1}}{I_{m+1,2}} - 1 \right) \Big] \end{aligned} \tag{13.148}$$

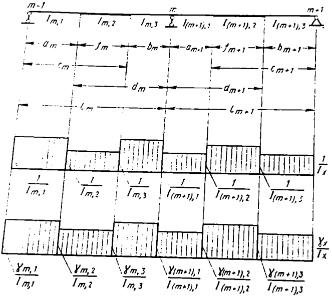


Fig. 13.45

$$\begin{aligned}A^1_{m,m+1} = {} & l_{m+1} \frac{I_C}{I_{(m+1),1}} \times \\ & \times \Big[1 + \frac{b_{m+1}^2}{l_{m+1}^2} \left(\frac{I_{(m+1),1}}{I_{(m+1),3}} - \frac{I_{(m+1),1}}{I_{(m+1),2}} \right) \times \\ & \times \left(3 - 2\frac{b_{m+1}}{l_{m+1}} \right) + \\ & + \frac{d_{m+1}^2}{l_{m+1}^2} \left(\frac{I_{(m+1),1}}{I_{(m+1),2}} - 1 \right) \left(3 - 2\frac{d_{m+1}}{l_{m+1}} \right) \Big] \end{aligned} \tag{13.149}$$

$$A^2_{m,\,m-1} = -6\frac{I_C}{l_m}\left(\gamma_{m,\,1}\frac{a_m}{I_{m,\,1}} + \right.$$
$$\left. + \gamma_{m,\,2}\frac{f_m}{I_{m,\,3}} + \gamma_{m,\,3}\frac{b_m}{I_{m,\,3}}\right) \tag{13.150}$$

$$A^2_{m,\,m} = 6\frac{I_C}{l^2_m}\left(\gamma_{m,\,1}\frac{a_m}{I_{m,\,1}} + \gamma_{m,\,2}\frac{f_m}{I_{m,\,2}} + \gamma_{m,\,3}\frac{b_m}{I_{m,\,3}}\right) +$$
$$+ 6\frac{I_C}{l^2_{m+1}}\left(\gamma_{(m+1),\,1}\frac{a_{m+1}}{I_{(m+1),\,1}} + \right.$$
$$\left. + \gamma_{(m+1),\,2}\frac{f_{m+1}}{I_{(m+2),\,2}} + \gamma_{(m+1),\,3}\frac{b_{m+1}}{I_{(m+1),\,3}}\right) \tag{13.151}$$

$$A^2_{m,\,m+1} = -6\frac{I_C}{l^2_{m+1}}\left(\gamma_{(m+1),\,1}\frac{a_{m+1}}{I_{(m+1),\,1}} + \gamma_{(m+1),\,2}\frac{f_{m+1}}{I_{(m+2),\,2}} + \right.$$
$$\left. + \gamma_{(m+1),\,3}\frac{b_{m+1}}{I_{(m+1),\,3}}\right) \tag{13.152}$$

Let us examine the influence of supports sagging on the bending moments.

If supports of an initially straight beam sag so that they are not on one straight line in the final state, then the beam is in flexure.

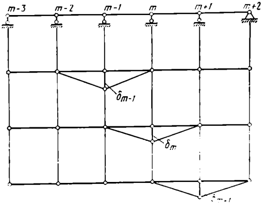


Fig. 13.46

The only external forces acting upon the beam in this case are the vertical support reactions; therefore the bending moment diagram will consist only of straight lines between the supports (without irregularities at supports).

Let us examine the sagging of supports $(m-1)$, m, and $(m+1)$. Coefficients $A_{m,\,i}$ will be determined by the formulas derived above, only the coefficients $\overline{\Pi}_{m-2}$, $\overline{\Pi}_{m-1}$, $\overline{\Pi}_{m}$, $\overline{\Pi}_{m+1}$, $\overline{\Pi}_{m+2}$ change; they may be written, according to Fig. 13.46, as

$$\overline{\Pi}_{m-2} = -6EI_C\frac{\delta_{m-1}}{l_{m-1}} = -\eta_{m-1,\,1} \tag{13.153}$$

$$\begin{aligned}\overline{\Pi}_{m-1} &= 6EI_C\left(\frac{\delta_{m-1}}{l_{m-1}} + \frac{\delta_{m-1}}{l_m}\right) - 6EI_C\frac{\delta_m}{l_m} = \\ &= (\eta_{m-1,\,1} + \eta_{m-1,\,2}) - \eta_{m,\,1}\end{aligned} \tag{13.154}$$

$$\begin{aligned}\overline{\Pi}_{m} &= 6EI_C\left(\frac{\delta_m}{l_m} + \frac{\delta_m}{l_{m+1}}\right) - 6EI_C\frac{\delta_{m-1}}{l_m} - \\ &\quad - 6EI_C\frac{\delta_{m+1}}{l_{m+1}} = \\ &= -\eta_{m-1,\,2} + (\eta_{m,\,1} + \eta_{m,\,2}) - \eta_{m+1,\,1}\end{aligned} \tag{13.155}$$

$$\begin{aligned}\overline{\Pi}_{m+1} &= 6EI_C\left(\frac{\delta_{m+1}}{l_{m+1}} + \frac{\delta_{m+1}}{l_{m+2}}\right) - 6EI_C\frac{\delta_m}{l_{m+1}} = \\ &= -\eta_m + (\eta_{m+1,\,1} + \eta_{m+1,\,2})\end{aligned} \qquad 13.156)$$

$$\overline{\Pi}_{m+2} = -6EI_C\frac{\delta_{m+1}}{l_{m+1}} = -\eta_{m+1,\,1} \tag{13.157}$$

where

$$\left.\begin{aligned}\eta_{m-1,\,1} &= 6EI_C\frac{\delta_{m-1}}{l_{m-1}}, \quad & \eta_{m-1,\,2} &= 6EI_C\frac{\delta_{m-1}}{l_m} \\ \eta_{m,\,1} &= 6EI_C\frac{\delta_m}{l_m}, \quad & \eta_{m,\,2} &= 6EI_C\frac{\delta_m}{l_{m+1}} \\ \eta_{m+1,\,1} &= 6EI_C\frac{\delta_{m+1}}{l_{m+1}}, \quad & \eta_{m+1,\,2} &= 6EI_C\frac{d_{m+1}}{l_{m+2}}\end{aligned}\right\} \tag{13.158}$$

When other supports sag as well, the problem is solved by superposition of results obtained for separate supports.

Consider the bending moments caused by rotation of a firmly fixed support. The bending moment diagram of the beam will con-

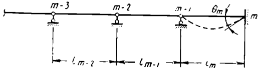


Fig. 13.47

sist of straight lines between the supports. In this case it is only necessary to determine coefficients $\overline{\Pi}$. If support m is fixed (Fig. 13.47), then

$$\left.\begin{aligned} &\overline{\Pi}_{m-2} = \overline{\Pi}_{m-1} = 0 \\ &\overline{\Pi}_m = 6EI_C\Theta_m = \lambda_m \\ \text{where} \qquad &\lambda_m = 6EI_C\Theta_m \end{aligned}\right\} \qquad (13.159)$$

Equation numbers $(m + 1)$ and $(m + 2)$ are missing in the system of equations.

Sagging and rotation of supports may be used in various problems to reduce the design moments.

Let us now examine the influence of uneven heating of the top and bottom surfaces of a beam.

Suppose the beam has an invariable cross section within each span and the temperature in the m-th span (Fig. 13.48a) changes linearly (Fig. 13.48b). Angle of rotation of two adjacent sections in an examined portion of the beam (Fig. 13.48a) is expressed by the following formula

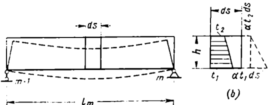


Fig. 13.48

$$d\gamma = \frac{(\alpha t_1 - \alpha t_2)}{h_m} = \frac{\alpha \Delta t}{h_m}\, ds \qquad (13.160)$$

The end sections of the m-th span will form an angle

$$\beta_m = \int_0^{l_m} \frac{\alpha \Delta t}{h_m}\, ds = \frac{\alpha \Delta t}{h_m}\, l_m \qquad (13.161)$$

due to heating, each end turning to an angle $\frac{\beta_m}{2}$. Thus, we finally obtain

$$\Pi_{m-1} = \Pi_m = 6EI_C \frac{\alpha \Delta t}{2h_m}\, l_m = 3EI_C \frac{\alpha \Delta t}{h_m}\, l_m = \varphi_m \qquad (13.162)$$

where

$$\varphi_m = 3EI_C\alpha \frac{l_m \Delta t}{h_m} \tag{13.163}$$

$$\Delta t = t_1 - t_2$$

and α is the coefficient of linear expansion of the material.

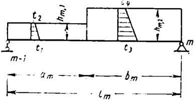


Fig. 13.49

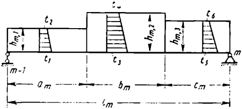


Fig. 13.50

The following formulas are true for a beam with two different linear temperature patterns within each span (Fig. 13.49)

$$\beta_m = \int_0^{a_m} \frac{\alpha \Delta t_1}{h_{m,1}}\, ds + \int_0^{b_m} \frac{\alpha \Delta t_2}{h_{m,2}}\, ds = \frac{\alpha \Delta t_1}{h_{m,1}} a_m + \frac{\alpha \Delta t_2}{h_{m,2}} b_m \tag{13.164}$$

hence,

$$\Pi_{m-1} = \Pi_m = \varphi_m \tag{13.165}$$

where

$$\varphi_m = 3EI_C\alpha \left(\Delta t_1 \frac{a_m}{h_{m,1}} + \Delta t_2 \frac{b_m}{h_{m,2}}\right) \tag{13.166}$$

$$\Delta t_1 = t_1 - t_2; \qquad \Delta t_2 = t_3 - t_4 \tag{13.167}$$

Similarly, for three temperature patterns (Fig. 13.50)

$$\Pi_{m-1} = \Pi_m = \varphi_m \tag{13.168}$$

where

$$\varphi_m = 3EI_C\alpha\left(\Delta t_1 \frac{a_m}{h_{m,1}} + \Delta t_2 \frac{b_m}{h_{m,2}} + \Delta t_3 \frac{c_m}{h_{m,3}}\right) \qquad (13.169)$$

$$\Delta t_1 = t_1 - t_2; \quad \Delta t_2 = t_3 - t_4; \quad \Delta t_3 = t_5 - t_6 \qquad (13.170)$$

On the basis of the above dependencies we may compile formulas for frames having members of variable sections.

Let us consider the example of a П-shaped frame with fixed ends (Fig. 13.51).

First of all, a point should be made on the choice of a principal model for the work method. Such a model should be chosen so that it must suit all (or the most) of the support conditions possible within the given geometry of the axis of the frame. This will make it possible to use the coefficients obtained for a frame with fixed ends in designing frames with different support conditions, but with the same geometry and loading. All these requirements are met by the basic model presented in Fig. 13.51*f*, because it may be used to solve problems given in Figs. 13.51*a-e*.

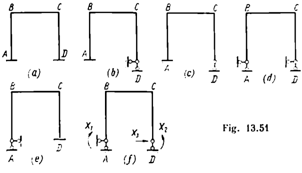


Fig. 13.51

Support moments and a horizontal component of the support D reaction multiplied by the height of the frame h (Fig. 13.52) are taken as the unknowns

$$M_A = X_1; \quad M_D = X_2; \quad hH_D = X_3 \qquad (13.171)$$

The canonical equations may be written as

$$\left.\begin{aligned} \delta_{11}X_1 + \delta_{12}X_2 + \delta_{13}X_3 + \Delta_{1p} = 0 \\ \delta_{21}X_1 + \delta_{22}X_2 + \delta_{23}X_3 + \Delta_{2p} = 0 \\ \delta_{31}X_1 + \delta_{32}X_2 + \delta_{33}X_3 + \Delta_{3p} = 0 \end{aligned}\right\} \qquad (13.172)$$

Multiplying all the coefficients of (13.172) by $\frac{6EI_2}{l}$ and denoting each by

$$\left.\begin{aligned}
&6\,\frac{\delta_{11}EI_2}{l}=A_1;\quad 6\,\frac{\delta_{22}EI_2}{l}=A_2;\quad 6\,\frac{\delta_{33}EI_2}{l}=A_3\\
&6\,\frac{\delta_{12}EI_2}{l}=6\,\frac{\delta_{21}EI_2}{l}=A_4;\quad 6\,\frac{\delta_{13}EI_2}{l}=6\,\frac{\delta_{31}EI_2}{l}=A_5\\
&6\,\frac{\delta_{23}EI_2}{l}=6\,\frac{\delta_{32}EI_2}{l}=A_6;\quad 6\,\frac{\Delta_{1p}EI_2}{l}=\Pi_1\\
&6\,\frac{\Delta_{2p}EI_2}{l}=\Pi_2;\quad 6\,\frac{\Delta_{3p}EI_2}{l}=\Pi_3
\end{aligned}\right\}\quad(13.173)$$

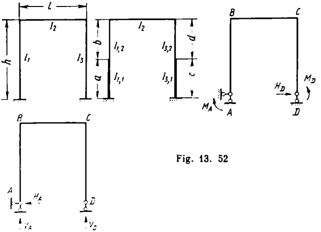


Fig. 13. 52

we obtain

$$\left.\begin{aligned}
A_1X_1+A_4X_4+A_5X_5&=-\Pi_1\\
A_4X_1+A_2X_2+A_6X_3&=-\Pi_2\\
A_5X_1+A_6X_2+A_3X_3&=-\Pi_3
\end{aligned}\right\}\quad(13.174)$$

Coefficients A_i are determined by unit factors (Fig. 13.53)

$$A_1=6\,\frac{I_2}{l}\left[\int_0^a\frac{dx}{I_{11}}+\int_a^h\frac{dx}{I_{12}}+\int_0^l\frac{x^2}{I_2l^2}\,dx\right]=$$

$$=2+6\left(\frac{I_2a}{I_{11}l}+\frac{I_2b}{I_{12}l}\right)$$

$$A_2=6\frac{I_2}{l}\left[\int_0^c \frac{dx}{I_{31}}+\int_c^h \frac{dx}{I_{32}}+\int_0^l \frac{x^2}{I_2l^2}dx\right]=$$

$$=2+6\left(\frac{I_2c}{I_{31}l}+\frac{I_2d}{I_{31}l}\right)$$

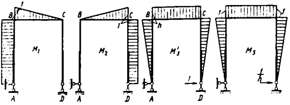


Fig. 13.53

$$A_3=6\frac{I_3}{l}\left[\int_0^a \frac{x^2\,dx}{h^2I_{11}}+\int_a^h \frac{x^2\,dx}{h^2I_{12}}+\int_0^c \frac{x^2\,dx}{h^2I_{31}}+\right.$$

$$\left.+\int_c^h \frac{x^2\,dx}{h^2I_{32}}+\int_0^l \frac{dx}{I_2}\right]=6+2\frac{hI_2}{lI_{12}}\times$$

$$\times\left[1+\frac{a^3}{h^3}\left(\frac{I_{12}}{I_{11}}-1\right)\right]+$$

$$+2\frac{hI_2}{lI_{32}}\left[1+\frac{c^3}{h^3}\left(\frac{I_{32}}{I_{31}}-1\right)\right];\quad A_4=1$$

$$A_5=6\frac{I_2}{l}\left[\int_0^a \frac{x\,dx}{hI_{11}}+\int_a^h \frac{x\,dx}{hI_{12}}+\int_0^l \frac{x\,dx}{lI_2}\right]=$$

$$=3+3\frac{hI_2}{lI_{12}}\left[1+\frac{a^2}{h^2}\left(\frac{I_{12}}{I_{11}}-1\right)\right]$$

$$A_6=6\frac{I_2}{l}\left[\int_0^c \frac{x\,dx}{hI_{31}}+\int_c^h \frac{x\,dx}{hI_{32}}+\int_0^l \frac{x\,dx}{lI_2}\right]=$$

$$=3+3\frac{hI_2}{lI_{32}}\left[1+\frac{c^2}{h^2}\left(\frac{I_{32}}{I_{31}}-1\right)\right]$$

Coefficients Π_1, Π_2, Π_3 depend on the loads.

Examine the loading of the bottom section of the left strut of the frame (Fig. 13.54).

The following notations are introduced:

M_0, M_1, M_2, M_3 = moments of the zero, first, second and third orders respectively about point A

$\overline{M}_0$, $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$ = moments of the zero, first, second and third orders respectively about point B

Relationships similar to (13.84)-(13.87) exist between the moments which are taken relative to points A and B.

Let

$$M_0h=\Delta_0;\quad M_1=\Delta_1;\quad \frac{M_2}{h}=\Delta_2;\quad \frac{M_3}{h^2}=\Delta_3 \qquad (13.175)$$

and $dP=q_x\,dx$.

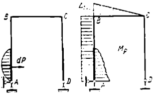


Fig. 13.54

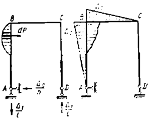


Fig. 13.55

Expressions for support reactions (in the principal model) from the applied frame loading are written as

$$\sum M_A=0,\quad V_Dl-\int_0^a xq_x\,dx=V_Dl-M_1=0$$

hence

$$V_D=\frac{\Delta_1}{l};\quad V_A=-V_D=-\frac{\Delta_1}{l}$$

$$\sum X=0;\quad H_A-\int_0^a q_x\,dx=0$$

from which

$$H_A=\frac{\Delta_0}{h}$$

Coefficients Π_1, Π_2, Π_3 are expressed in higher moments of the load which is applied to the bottom section of the strut with respect to point A

$$\Pi_1=A_1\Delta_1-\frac{6I_2}{l}\int_0^a \frac{M_P}{I_{11}}\,dx=A_1\Delta_1-\frac{6I_2}{I_{11}l}\cdot\frac{M_2}{2!}=$$

$$=A_1\Delta_1-3\,\frac{hI_2}{lI_{11}}\Delta_2 \qquad (13.176)$$

$$\Pi_2=A_4\Delta_1=\Delta_1 \qquad (13.177$$

$$\Pi_3 = A_5\Delta_1 - \frac{6I_2}{l}\int_0^a \frac{M_p x}{I_{11}h}\,dx = A_5\Delta_1 - \frac{6I_2}{lhI_{11}} - $$
$$- \frac{M_3}{3!} = A_5\Delta_1 - \frac{hI_2}{lI_{11}}\Delta_3 \tag{13.178}$$

Examine the loading of the top section of the left strut (Fig. 13.55). The support reactions are the same as in the case of the bottom section loading.

Coefficients Π_1, Π_2, Π_3 are expressed in higher order moments of the top section loading with respect to point A; these expressions

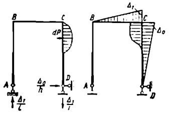


Fig. 13.56

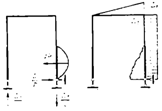


Fig. 13.57

are derived using the relation between moments relative to points A and B, established above

$$\Pi_1 = (A_5 - 3)\,\Delta_0 - 6\,\frac{I_2\Delta_1 l}{l3I_2} - \frac{6I_2}{l}\int_0^b \frac{M_p}{I_{12}}\,dx =$$
$$= (A_5 - 3)\,\Delta_0 - 2\Delta_1 - \frac{6I_2}{lI_{12}}\cdot\frac{\overline{M}_2}{2!} = (A_5 - 3)\,\Delta_0 -$$
$$- 2\Delta_1 - 3\,\frac{hI_2}{lI_{12}}\,(\Delta_0 - 2\Delta_1 + \Delta_2) \tag{13.179}$$

$$\Pi_2 = A_4\Delta_1 = \Delta_1 \tag{13.180}$$

$$\Pi_3 = \frac{6I_2}{l}\Big[\int_0^a \Delta_0\,\frac{x^2}{h^2I_{11}}\,dx + \int_a^h \Delta_0\,\frac{x^2}{h^2I_{12}}\,dx + \frac{\Delta_1 l}{2I_2} - \int_0^b\Big(1 - \frac{x}{h}\Big)\frac{M_p}{I_{11}}\,dx\Big] =$$
$$= \frac{hI_2}{lI_{12}}\Big[1 + \frac{a^3}{h^3}\Big(\frac{I_{12}}{I_{11}} - 1\Big)\Big]\Delta_0 + 3\Delta_1 - \frac{6I_2}{lI_{12}}\Big(\frac{\overline{M}_2}{2!} - \frac{\overline{M}_3}{3!\,h}\Big) =$$
$$= \frac{hI_2}{lI_{12}}\Big[1 + \frac{a^3}{h^3}\Big(\frac{I_{12}}{I_{11}} - 1\Big)\Big]\Delta_0 +$$
$$+ 3\Delta_1 - \frac{hI_2}{lI_{12}}\,(2\Delta_0 - 3\Delta_1 + \Delta_3) \tag{13.181}$$

The right strut loadings are solved similarly (Figs. 13.56 and 13.57). However, it is more advisable to take the horizontal component of the support reaction A, multiplied by h, for one of the unknowns. Then the coefficients A_i will not change and the solution of the system will be valid, if $X_3 = hH_A$.

When gantry beam BC (Fig. 13.58) is loaded, the higher order moments are evaluated relative to point B and the following designations are used:

$$\Delta_0 = M_0 l; \quad \Delta_1 = M_1; \quad \Delta_2 = \frac{M_2}{l}; \quad \Delta_3 = \frac{M_3}{l^2} \qquad (13.182)$$

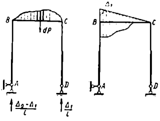


Fig. 13.58

Expressions for the support reactions for the given gantry loading are derived from the equilibrium equations referring to the principal model

$$\sum M_A = 0; \quad V_D l - \int_0^l x q_x \, dx = V_D l - M_1 = 0$$

hence,

$$V_D = \frac{\Delta_1}{l}; \qquad \sum M_B = 0$$

$$V_A l - \int_0^l q_x (l - x) \, dx = 0$$

$$V_A = \frac{\Delta_0 - \Delta_1}{l}; \quad H_A = 0$$

Coefficients Π_1, Π_2, Π_3 are determined in terms of higher moments of the loading applied to the gantry beam BC, with respect to point B:

$$\Pi_1 = \frac{6I_2}{l} \left[\int_0^l \frac{\Delta_1}{I_2} \left(1 - \frac{x}{l}\right)^2 dx - \int_0^l \frac{1}{I_2} \left(1 - \frac{x}{l}\right) M_p \, dx \right] =$$

$$= \frac{6I_2}{l} \left[\frac{\Delta_1 l}{3I_2} - \frac{l}{I_2} \left(\frac{M_2}{2!\, l} - \frac{M_3}{3!\, l^2} \right) \right] = 2\Delta_1 - 3\Delta_2 + \Delta_3 \qquad (13.183)$$

$$\Pi_2 = \frac{6I_2}{l} \left[\int_0^l \frac{\Delta_1}{I_2} \left(1 - \frac{x}{l}\right) \frac{x}{l} \, dx - \int_0^l \frac{M_p x}{I_2 l} \, dx \right] =$$

$$= \frac{6I_2}{l} \left(\frac{l}{6I_2} \Delta_1 - \frac{1}{I_2 l} \cdot \frac{M_3}{3!} \right) = \Delta_1 - \Delta_3 \qquad (13.184)$$

$$\Pi_3=\frac{6I_2}{l}\left[\frac{\Delta_1}{I_2}\left(1-\frac{x}{l}\right)dx-\int_0^l\frac{M_p}{I_2}dx\right]=$$

$$=\frac{6I_2}{l}\left(\frac{l}{2I_2}\Delta_1-\frac{1}{I_2}\cdot\frac{M_2}{2!}\right)=3\left(\Delta_1-\Delta_2\right) \qquad (13.185)$$

Frames with continuously variable sections are solved similarly if the loaded members consist of invariable or stepped sections.

The method outlined here was used in deriving formulas for beams and frames with members of variable sections. These formulas are of a general type, and it is possible to use them to determine reference values for almost any load, because the computations of the higher moments used as functional solutions are always easy to carry out. In the works of N. Valishvili 119 loading schemes are examined and the moments of the first three orders are calculated for these schemes. These references contain formulas for bending moments, transversal forces, rotations and deflections of single beams of invariable sections within the span under the influence of the same loading schemes; all the formulas use the notation of higher moments' symbolics. After the redundancy of structures is solved, these formulas may be used to plot the summary diagrams of bending moments and transversal forces for beams and frames of both invariable and variable cross sections.

14. RIGIDITY OF BEAMS COMPOSED OF PRISMATIC PARTS

Let us evaluate the variable rigidity of beams, which remains constant within separate sections along the beam and changes only from one section to the other. In such cases the beam has a step form along its length (Fig. 14.1*a*).

The following method of determining angular and linear displacements, suggested by B. Zhemochkin, is based on replacing the stepped beam by a beam of constant stiffness. Displacements of such a beam, both linear and angular, will be equal to the displacements of the stepped beam, if the effective rigidity is calculated according to certain rules.

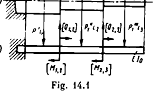


Fig. 14.1

The given beam may have an arbitrary number of parts with different rigidity. The transverse loading applied to the beam may be also arbitrary. To simplify the drawing, only one point load is applied to each part of the stepped beam shown in Fig. 14.1.

Let P_1', P_1'', P_1''' = point loads applied to the first, second and third sections, respectively

EI_1, EI_2, EI_3 = rigidity of the beam within the first, second and third sections, respectively

EI_0 = arbitrary effective rigidity, i.e., rigidity of the constant section beam replacing the stepped beam

and

$$\frac{EI_0}{EI_1} = i_1; \quad \frac{EI_0}{EI_2} = i_2; \quad \frac{EI_0}{EI_3} = i_3 \tag{14.1}$$

Linear and angular displacements of a beam of effective rigidity will correspond to the ones of the given beam if the given forces P_1', P_1'', P_1''', . . . are replaced by effective forces $(P_1' \cdot i_1)$, $(P_1'' \cdot i_2)$, $(P_1''' \cdot i_3)$, . . . applied to the same sections; in addition, fictitious loads and force couples must be applied at the positions of the changing sections. These fictitious loads are

$$\left.\begin{aligned} [Q_{1.2}] &= Q_{1.2}(i_2 - i_1); \quad [M_{1.2}] = M_{1.2}(i_2 - i_1) \\ [Q_{2.3}] &= Q_{2.3}(i_3 - i_2); \quad [M_{2.3}] = M_{2.3}(i_3 - i_2) \end{aligned}\right\} \tag{14.2}$$

etc., for the other sections.

$[Q_{1.2}]$ and $[M_{1.2}]$ are additional shear force and point bending moment applied at the joint between the first and the second sections; $[Q_{2.3}]$ and $[M_{2.3}]$ are similar quantities between the second and the third sections, etc.

$Q_{1.2}$, $M_{1.2}$, $Q_{2.3}$, $M_{2.3}$ denote factual shear forces and bending moments, which exist at the joints of the sections of different rigidity due to the given forces P_1', P_1'', P_1''', . . . applied to the beam.

Figure 14.1*b* presents the beam of constant effective rigidity with all its loads. Rotations and deflections of this beam are equal to the rotations and deflections of the stepped beam of variable rigidity.

Zhemochkin proved the effectiveness of the higher moments' theory by applying it to such problems as determining beam displacements.

The method suggested by Zhemochkin is quite efficient in all cases of compound loads applied to beams with constant rigidity along the separate sections, that is, when the beam is built up of several prismatic parts.

For example, consider a three-section beam (Fig. 14.2*a*) under the influence of distributed deflecting loads (Fig. 14.2*b*); let us examine the transforms necessary to provide a beam of effective constant rigidity.

This problem will be solved if the following fictitious intersecting forces and bending moments are applied to boundary points

$$[Q_{1.2}] = Q_{1.2}(i_2 - i_1) \quad \text{and} \quad [Q_{2.3}] = Q_{2.3}(i_3 - i_2) \tag{14.3}$$

$$[M_{1.2}] = M_{1.2}(i_2 - i_1) \quad \text{and} \quad [M_{2.3}] = M_{2.3}(i_3 - i_2) \tag{14.4}$$

In addition, the distributed loads must be transformed without moving them along the beam. This means that a uniformly distributed load of intensity q will transform to three different uniformly distributed loads (qi_1), (qi_2), and (qi_3) acting within the first, second and third sections, respectively; a linearly increasing distributed load will be replaced by two different increasing loads, one within the second section from 0 to (ka) i_2 and the other, within the third section from (ka) i_3 to k $(a + b)$ i_3 instead of the given load increasing from (ka) to k $(a + b)$.

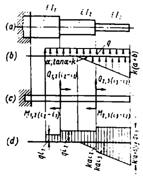


Fig. 14.2

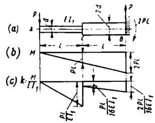


Fig. 14.3

Zhemochkin's method of solving problems in flexure of structures will be shown later, the examples being elucidated by the graphical-analytical method.

Diagrams of bending moments M (Fig. 14.3*b*) and curvatures $k = \frac{M}{EI_1}$ (Fig. 14.3*c*) are plotted for the beam shown in Fig. 14.3*a*. The diagram of curvatures allows for the different rigidities of the first and second sections of the beam. If the rigidity of the first section is denoted by EI_1, then the rigidity of the second section will be $16EI_1$, because its diameter is twice the diameter of the first section.

Displacements of a beam of constant effective rigidity EI_0 are determined according to the following rule: *the given beam and the beam of effective rigidity have equal linear and angular displacements, if the curvature diagrams in both cases are identical; the displacement diagrams are integral with respect to the diagram k.*

Let us choose an arbitrary rigidity EI_0 of the "effective" beam, say $EI_0 = EI_1$ and assume that its curvature diagram k (Fig. 14.4*d*) is identical to the curvature diagram shown in Fig. 14.3*c*. Such a diagram may be obtained for a beam of constant rigidity, if an appropriate load (theoretical) is applied.

For the given beam we take the curvature diagram k; according to Fig. 14.4*d* we solve the problem of construction of the beam loading scheme by successfully moving from the curvature diagram k to the bending moment diagram M, then to diagrams Q and q using the differential relationships between q, Q and M.

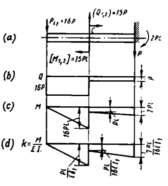


Fig. 14.4

The bending moment diagram of a beam of constant rigidity is obtained by multiplying the ordinates of the k diagram (Fig. 14.4*d*) by the effective rigidity $EI_0 = EI_1$; this results in $\frac{Pl}{16EI_0}\,EI_1 = \frac{Pl}{16}$ instead of $\frac{Pl}{16EI_1}$ and $\frac{2Pl}{16EI_0}\,EI_1 = \frac{2Pl}{16}$ instead of $\frac{2Pl}{16EI_1}$. These ordinates are used in constructing the bending moment diagram (Fig. 14.4*c*) for the beam of constant rigidity.

The Q diagram (Fig. 14.4*b*) is plotted after its ordinates are determined as slope tangents of the M diagram (Fig. 14.4*c*). Within the first section of the beam $Q = \tan\alpha_1 = -\frac{Pl}{l} = -P$; within the second section $Q = \tan\alpha_2 = -\frac{Pl}{16}\cdot\frac{1}{l} = -\frac{P}{16}$. When $Q = \text{const}$ within each section of the beam, $q = 0$, which means that the "effective" beam cannot have distributed loads.

According to the Q diagram (Fig. 14.4*b*), a point load P is to be applied downwards at point A of the loading diagram (Fig. 14.4*a*), a force $\frac{15}{16}P$ applied upwards at point C, and a force $\frac{P}{16}$ applied downwards at point B. In addition, a force couple $M = \frac{15}{16}Pl$ is to be applied at point C, which is equal to the step in the bending moment diagram M (Fig. 14.3*c*).

Thus, if a beam is under the influence of the load shown in Fig. 14.4*a*, then the problem is solved.

Now, after both the given beam and the given load are retraced to give an "effective" beam and effective load, thus proving the conversion method, we shall examine formulas which make it possible to solve the problem directly, without plotting the diagrams.

The fictitious load is determined according to the following rules. First, reduction coefficients are found for the n-th section of the beam

$$i_n = \frac{EI_0}{EI_n}$$

where EI_0 = constant arbitrary rigidity for all beam sections
EI_n = rigidity of the n-th section of the given beam

In our example, for the left (first) section of the beam

$$i_1 = \frac{EI_0}{EI_1} = \frac{EI_1}{EI_1} = 1$$

and for the second section

$$i_2 = \frac{EI_0}{EI_2} = \frac{EI_1}{16EI_1} = \frac{1}{16}$$

Next, a fictitious load is applied to the beam at the joint of sections n and $(n+1)$ of different rigidity; this load consists of

(1) point force couple with a moment $[M_{n,\,n+1}] = M_{n,\,n+1}\cdot(i_{n+1}-i_n)\,n$

(2) transverse force $[Q_{n,\,n+1}] = Q_{n,\,n+1}(i_{n+1}-i_n)$, in which $M_{n,\,n+1}$ and $Q_{n,\,n+1}$ are the actual bending moment and shear force, respectively, appearing at the joint of sections n and $(n+1)$ of the given beam.

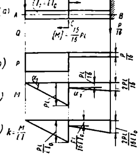


Fig. 14.5

For the problem shown in Figs. 14.3 and 14.4

$$[M_{1,2}] = M_{1,2}(i_2-i_1) = (-Pl)\left(-\frac{15}{16}\right) = \frac{15}{16}Pl$$

$$[Q_{1,2}] = Q_{1,2}(i_2-i_1) = (-P)\left(-\frac{15}{16}\right) = \frac{15}{16}P$$

If $EI_0 = 16EI_1$ is taken for the effective rigidity, then a solution shown in Fig. 14.5 will be obtained.

The curvature diagram k (Fig. 14.5d) is taken for the initial diagram in this case. The bending moment diagram M is obtained by multiplying the k-ordinates by the effective rigidity $EI_0 = 16EI_1$ (Fig. 14.5c), etc.

Zhemochkin's formulas give

$$i_1 = \frac{EI_0}{EI_1} = \frac{16EI_1}{EI_1} = 16; \quad i_2 = \frac{EI_0}{EI_2} = 1$$

Hence, the fictitious force factors at the mid beam section are

$$[Q_{1,2}] = Q_{1,2}(i_2 - i_1) = (-P)(1-16) = 15P$$
$$[M_{1,2}] = M_{1,2}(i_2 - i_1) = (-P)(1-16) = 15Pl$$

As the rotation angle is zero when $x = 2l$ for the beam of effective rigidity, then

$$0 = EI\Theta_0 + \frac{M_2}{2}$$

where $\frac{M_2}{2}$ is the second moment of the total load, divided by 2.

Hence, according to Fig. 14.5*a*

$$EI\Theta_0 = -\frac{M_2}{2} = -\left[(15Pl)\,l - 16P\frac{(2l)^2}{2} - 15P\frac{l^2}{2}\right] = \frac{19}{2}Pl^2$$

and, as deflection v is zero when $x = 2l$, then

$$0 = EIv_0 + EI\Theta_0\cdot 2l + \frac{M^3}{3!}$$

where $\frac{M_3}{3!}$ is the third moment of the total load, divided by 3!

Hence,

$$EIv_0 = -EI\Theta_0 2l - \frac{M_3}{3!} = -\frac{19}{2}Pl^2\cdot 2l -$$
$$-\left[(15Pl)\frac{l^2}{2} + 16P\frac{(2l)^3}{3!} + 15\frac{Pl^3}{3!}\right] = -\frac{23}{3}Pl^3$$

Zhemochkin's method can be applied to torsion and tension-and-compression of bars composed of several sections with different rigidity.

15. MOMENTS OF AREA OF FLEXIBILITY DIAGRAM FOR NON-UNIFORM BEAMS

The method outlined in the previous section cannot be used to evaluate beams of continuously varying rigidity. It seems possible to calculate the fictitious load on the basis of the bending moment diagram of the given beam of continuously varying rigidity, so that the "effective" beam may have the same bending moment diagram as the given beam. But this method will hardly give any practically useful solutions.

Let us examine other methods of solving the same problem.

Consider a beam under the influence of a compound load shown in Fig. 15.1. Let the left end be fixed so that the initial angle $\Theta_0 = 0$ and the initial deflection $v_0 = 0$. Then the bending moment at an arbitrary section X of the beam at a distance x from the origin of coordinates O is given as

$$M_X = \sum M (x-a)^0 + \sum P (x-b) + \\ + \sum q \frac{(x-c)^2}{2} + \sum k \frac{(x-d)^3}{3!} \qquad (15.1)$$

where the sum is evaluated over all the loads of the same type, each shown in Fig. 15.1 by one load.

Similarly, at any section U at a distance u from point O

$$M_U = \sum M (u-a)^0 + \sum P (u-b) + \sum q \frac{(u-c)^2}{2} + \sum k \frac{(u-d)^3}{3!} \quad (15.2)$$

Suppose section U is at the left of X, that is $u < x$.

The curvature k of the elastic curve at section U of the beam is given by $\frac{M_U}{EI_U}$, where EI_U is the variable rigidity of the beam. As

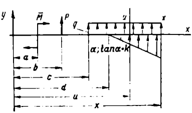


Fig. 15.1

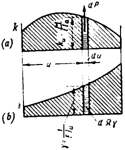


Fig. 15.2

EI_U and M_U are determined for every section of the beam, the curvature k_U may be calculated as well and the curvature diagram may be plotted as shown in Fig. 15.2*a*.

Using the graphical-analytical method to determine the linear and angular displacements in flexure by interpreting the diagram as a fictitious load $[P]$ within du, a differential of the load is given as

$$[dP] = k\,du = \frac{M_U}{EI_U}\,du \qquad (15.3)$$

If a diagram with the ordinates

$$\gamma U = \frac{1}{EI_U} \qquad (15.4)$$

is constructed (Fig. 15.2*b*) we have a diagram of a variable quantity γ inverse of beam transverse flexibility.

An element of the area of the Ω_γ-diagram is given by

$$d\Omega_\gamma = \gamma\, du = \frac{1}{EI_U}\, du \tag{15.5}$$

hence,

$$[dP] = M_U\, d\Omega_\gamma \tag{15.6}$$

Determining linear and angular displacements by the graphical-analytical method gives, as usual, the formulas

$$\theta_X = \theta_0 + \int\limits_{u=0}^{u=x} [dP] \tag{15.7}$$

$$v_X = v_0 + \theta_0 x + \int\limits_{u=0}^{u=x} [dP]\,(x-u) \tag{15.8}$$

Substitution of $[dP]$ in these formulas will yield the following expressions:

$$\theta_X = \theta_0 + \int\limits_{u=0}^{u=x} M_U\, d\Omega_\gamma \tag{15.9}$$

$$v_X = v_0 + \theta_0 x + \int\limits_{u=0}^{u=x} M_U\,(x-u)\, d\Omega_\gamma \tag{15.10}$$

where M_U is the bending moment of the total compound load taken at section U.

For example, deflections of a beam at section X are determined by substituting (15.2) in (15.10)

$$\begin{aligned} v_X = {} & v_0 + \theta_0 x + \sum \overline{M} \int\limits_{u=a}^{u=x} (u-a)^0 (x-u)\, d\Omega_\gamma + \\ & + \sum P \int\limits_{u=b}^{u=x} (u-b)(x-u)\, d\Omega_\gamma + \sum \frac{q}{2} \int\limits_{u=c}^{u=x} (u-c)^2 (x-u)\, d\Omega_\gamma + \\ & + \sum \frac{k}{3!} \int\limits_{u=d}^{u=x} (u-d)^3 (x-u)\, d\Omega_\gamma \end{aligned} \tag{15.11}$$

The first integral of this formula $\int\limits_{u=a}^{u=x} (u-a)^0 (x-u)\, d\Omega_\gamma$ has the multiplier $(u-a)^0 = 1$, which is artificially introduced so that all the integrals may have identical structures.

Let us examine the four integrals in the formula for v_X as biaxial moments of area Ω_γ.

Abscissas appearing in the expressions under the integral sign are shown in Fig. 15.3.

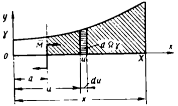


Fig. 15.3

According to Fig. 15.3, the first integral $\int\limits_{u=a}^{u=x} (u-a)^0 (x-u)\, d\Omega_\gamma$ is a biaxial static moment (i.e., M_1) of the flexibility diagram area with respect to two parallel vertical axes with abscissas x and a. The other integrals in the formula for v_X are similar biaxial moments of the flexibility diagram.

Biaxial moments are known to be composed in several ways, the two moments being of different orders within one biaxial moment; biaxial moments obtained above are

$$\int\limits_{u=a}^{u=x} (u-a)^0 (x-u)\, d\Omega_\gamma = M_{a^0x^1} \tag{15.12}$$

$$\int\limits_{u=b}^{u=x} (u-b)(x-u)\, d\Omega_\gamma = M_{b^1x^1} \tag{15.13}$$

$$\int\limits_{u=c}^{u=x} (u-c)^2 (x-u)\, d\Omega_\gamma = M_{c^2x^1} \tag{15.14}$$

$$\int\limits_{u=d}^{u=x} (u-d)^3 (x-u)\, d\Omega_\gamma = M_{d^3x^1} \tag{15.15}$$

where $M_{a^0x^1}$ is a biaxial moment of zero order with respect to the vertical axis with abscissa a, and of the first order with respect to the vertical axis with abscissa x; hence, the moment is of the first order; $M_{b^1x^1}$ is a biaxial moment of the first order with respect to two parallel vertical axes, one with abscissa b and the other with abscissa x, etc.

Substitution of biaxial moments in (15.11) will yield the following formula

$$v_X = v_0 + \Theta_0 x + \sum \overline{M} \cdot M_{a^0x^1} + \sum P M_{b^1x^1} + \\ + \sum q \frac{M_{c^2x^1}}{2} + \sum k \frac{M_{d^3x^1}}{3!} \tag{15.16}$$

Similarly, a formula for the angle Θ_X is derived by substituting $(u-a)^0$ or $(x-a)^0$ for the absent second multiplier, both equal to unity, in the integrand expressions, thus resulting in the generalization of expressions as in the case of the formula for v_X. This gives the following formula for the rotation angle

$$\Theta_X = \Theta_0 + \sum \overline{M} \cdot M_{a^0x^0} + \sum P M_{b^1x^0} + \\ + \sum q \frac{M_{c^2x^0}}{2!} + \sum k \frac{M_{d^3x^0}}{3!} \tag{15.17}$$

The expression for the angle Θ_X is a direct differential of the expression for v_X, in which the order of the moment with respect to the left vertical axis remains the same, whereas the order of the moment with respect to the other vertical axis x is a unit lower. The following expressions are derived for the area of the transverse flexibility diagram Ω_γ

$$M_{a^0x^0} = \int\limits_{u=a}^{u=x} (u-a)^0 (x-u)^0 \, d\Omega_\gamma \tag{15.18}$$

$$M_{b^1x^0} = \int\limits_{u=b}^{u=x} (u-b)(x-u)^0 \, d\Omega_\gamma \tag{15.19}$$

$$M_{c^2x^0} = \int\limits_{u=c}^{u=x} (u-c)^3 (x-u)^0 \, d\Omega_\gamma \tag{15.20}$$

$$M_{d^3x^0} = \int\limits_{u=d}^{u=x} (u-d)^3 (x-u)^0 \, d\Omega_\gamma \tag{15.21}$$

Thus, the formula for the area Ω_γ outlined by the elastic curve and the straight line (before deformation) of the beam axis may be constructed at once on the basis of a simple analogy, and this formula may be expressed in terms of biaxial moments, if required. For this it should be only necessary to consider that the area Ω_γ is integral with respect to the deflection v and all the orders of the moments must be increased by one when taken at abscissa x, i.e.,

$$\Omega_\gamma = v_0 x + \Theta_0 \frac{x^2}{2} \sum \overline{M} \frac{M_{a^0x^2}}{2} + \sum P \frac{M_{b^1x^2}}{2} + \\ + \sum q \frac{M_{c^2x^2}}{2!\ 2!} + \sum k \frac{M_{d^3x^2}}{3!\ 2!}, \text{ etc.} \tag{15.22}$$

It may be observed from comparison of formulas derived for Θ_X, v_X and Ω_γ, that as the integration is carried out with respect to the first index of the biaxial moment when passing from the left terms of the formulas to the right terms the factorials in the denominators change accordingly. If a term corresponding to a certain force factor is traced through formulas for Θ_x, v_x, Ω_γ, etc., then this sequence is accompanied by integration with respect to the second index of the biaxial moment. This means that it is advisable to calculate the biaxial moments of the flexibility diagram beforehand when a rigidity design of a beam of a continuously varying section is carried out. Orders of these moments are indicated in formulas (15.16) and (15.17), which are expressions for the angle Θ_x and the deflection v_x.

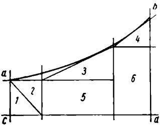


Fig. 15.4

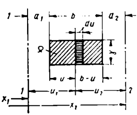


Fig. 15.5

These calculations are very simple to perform in practice, because the curved flexibility diagram may be replaced by a polygon with sufficient accuracy for all practical purposes, thus simplifying the calculations. For example, curve *ab* (Fig. 15.4) may be approximated by three tangents and the area below this broken line may be divided into a number of elementary figures, such as rectangles and triangles. One of the possible variants of such a case is shown in Fig. 15.4, the flexibility diagram area *cabd* being divided into triangles *1*, *2*, *3*, *4* and rectangles *5* and *6*.

The biaxial moments of triangles and rectangles are simple to operate with. To prove this, let us derive formulas for certain biaxial moments often used in practice.

(a) The area of a rectangle (Fig. 15.5).

Biaxial moments with respect to two parallel axes *1-1* and *2-2* with abscissas x_1 and x_2, respectively, are determined by the following formulas

$$M_{x_1^1 x_2^0} = \int^{\Omega} u_1 \, d\Omega_\gamma = \gamma \int_{u=0}^{u=b} (u + a_1) \, du =$$

$$= \gamma \left(\frac{b^2}{2} + a_1 b \right) = \Omega \left(\frac{b}{2} + a_1 \right) \tag{15.23}$$

$$M_{x_1^2x_2^0}=\int\limits^{\Omega} u_1^2\,d\Omega=\gamma\int\limits_{u=0}^{u=b}(u+a_1)^2\,du=\Omega\left(\frac{b^2}{3}+a_1b+a_1^2\right) \qquad (15.24)$$

$$M_{x_1^10x_2^1}=\int\limits^{\Omega} u_1u_2\,d\Omega=\gamma\int\limits_{u=0}^{u=b}(u+a_1)(b-u+a_2)\,du=$$

$$=\Omega\left(\frac{b^2}{6}+\frac{a_1b}{2}+\frac{a_2b}{2}+a_1a_2\right) \qquad (15.25)$$

$$M_{x_1^2x_2^1}=\int\limits^{\Omega} u_1^2u_2\,d\Omega=$$

$$=\Omega\left(\frac{b^3}{12}+\frac{a_1b^2}{3}+\frac{a_1^2b}{2}+\frac{a_2b^2}{3}+a_1a_2b+a_1^2a_2\right) \qquad (15.26)$$

(b) The area of a triangle with its apex pointing to the right (Fig. 15.6).

With respect to the same axes we find

$$M_{x_1^1x_2^0}=\int\limits^{\Omega} u_1\,d\Omega=\int\limits_{u=0}^{u=b}(a_1+u)\,\gamma_u\,du=$$

$$=\int\limits_{u=0}^{u=b}(a_1+u)\left(\gamma-\gamma\frac{u}{b}\right)du=\Omega\left(a_1+\frac{b}{3}\right) \qquad (15.27)$$

$$M_{x_1^2x_2^0}=\int\limits^{\Omega} u_1^2\,d\Omega=\int\limits_{u=0}^{u=b}(a_1+u)^2\,\gamma_u\,du=$$

$$=\Omega\left(a_1^2+\frac{2}{3}\,a_1b+\frac{b^2}{6}\right) \qquad (15.28)$$

$$M_{x_1^1x_2^1}=\int\limits^{\Omega} u_1u_2\,d\Omega=\Omega\left(\frac{b^2}{6}+a_1a_2+\frac{a_2b}{3}+\frac{2}{3}a_1b\right) \qquad (15.29)$$

$$M_{x_1^2x_2^1}=\int\limits^{\Omega} u_1^2u_2\,d\Omega=$$

$$=\Omega\left(\frac{2}{3}\,a_1^2b-\frac{a_1b^2}{3}+\frac{b^3}{15}+a_1^2a_2+\frac{2}{3}\,a_1a_2b+\frac{a_2b^2}{6}\right) \qquad (15.30)$$

As a rule, only angular and linear displacements are required when evaluating the rigidity of beams. They may be calculated by using formulas (15.17) and (15.16). Biaxial moments included in these formulas may be determined in two ways: from formulas (15.23)-(15.30) and from the general considerations on the higher moments given in Sec. 6-8.

Indeed, the higher moments in (15.17) are written as biaxial moments of the flexibility diagram, but they are essentially uniaxial moments of various orders and may be calculated according to the above mentioned rules concerning uniaxial moments.

Moments in (15.16) are more difficult to calculate because they are biaxial moments of various orders; however, due to the fact that all the moments in the right part of the formula are of the first order with respect to the right-hand bounding vertical line, they may be expressed in terms of uniaxial moments.

Suppose a biaxial moment of a figure Ω (Fig. 15.7) with respect to the bounding axis r-r and x-x, is to be determined, one of the moments (about the left axis) being of order n and the other (about the right axis), of the first order. Taking the element $d\Omega$ of this area with changing distances u and $(t-u)$ from the axes gives

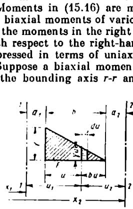


Fig. 15.6

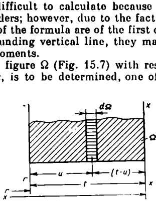


Fig. 15.7

$$M_{r^n x^1} = \int^{\Omega} u^n (t-u)\, d\Omega = t \int^{\Omega} u^n\, d\Omega - \int^{\Omega} u^{n+1}\, d\Omega =$$
$$= tM_{r^n} - M_{r^{n+1}} \tag{15.31}$$

i.e., the unknown biaxial moment is expressed in terms of uniaxial moments of orders n and $n+1$.

In particular cases, assuming $n = 0, 1, 2, 3, \ldots$, the following expressions are derived for substitution in formula (15.16)

$$\left.\begin{aligned}
M_{a^0x^1} &= (x-a)\,\Omega - M_{a^1}\\
M_{b^1x^1} &= (x-b)\,M_{b^1} - M_{b^2}\\
M_{c^2x^1} &= (x-c)\,M_{c^2} - M_{c^3}\\
M_{d^3x^1} &= (x-d)\,M_{d^3} - M_{d^4}, \text{ etc.}
\end{aligned}\right\} \tag{15.32}$$

When $n \geqslant 2$, the formulas of the relationships between biaxial and uniaxial moments relative to the right-hand vertical become more complicated, than those in expressions (15.31) and (15.32).

Calculation Examples

Problem 1. Design of a spindle of the front mandrel of a universal cylinder-grinding machine.

For the case of a continuously changing transverse flexibility, for example, when evaluating conically shaped members, the γ diagram is outlined by a parabola. The most suitable approximation of this shape is performed by dividing the area into triangles (Fig. 15.8). A similar method was used when calculating the rigidity of a spindle head of the front mandrel of a universal grinding machine.

Design formulas for a cone under the action of the loading shown in Fig. 15.9 (point loads and a uniformly distributed load) may be written as

$$\Theta_X = \Theta_0 + \sum_1^k P_k M_{a_k^1 x^0} + \frac{q}{2} M_{c_0^1 x^0} \tag{15.33}$$

$$v_X = v_0 + \Theta_0 x + \sum P_k M_{a_k^1 x^1} + \frac{q}{2} M_{c_0^2 x^1} \tag{15.34}$$

where $M_{a_k^1 x^0}$ = biaxial moment of area Ω_γ between a_k and x (of the first order with respect to the section with abscissa a_k and of zero order with respect to the section with abscissa x)

$M_{c_0^1 x^0}$ = biaxial moment of area Ω_γ between c_0 and x (of the second order with respect to the section with abscissa c_0 and of zero order with respect to the section with abscissa x), etc.

For a triangle with its apex pointing to the right (Fig. 15.10)

$$M_{a_k^1 x^0} = \Omega_\gamma \left(a_1 + \frac{b}{3} \right) \tag{15.35}$$

$$M_{c_0^2 x^0} = \Omega_\gamma \left(a_1^2 + \frac{2}{3} a_1 b + \frac{b^2}{6} \right) \tag{15.36}$$

$$M_{a_k^1 x^1} = \Omega_\gamma \left(\frac{b^2}{6} + a_1 a_2 - \frac{a_2 b}{3} + \frac{2}{3} a_1 b \right) \tag{15.37}$$

$$M_{c_0^2 x^1} = \Omega_\gamma \left(\frac{2}{3} a^2 b - \frac{a_1 b^2}{3} + \frac{b^3}{15} + a_1^2 a_2 + \frac{2}{3} a_1 a_2 b + \frac{a_2 b^2}{6} \right) \tag{15.38}$$

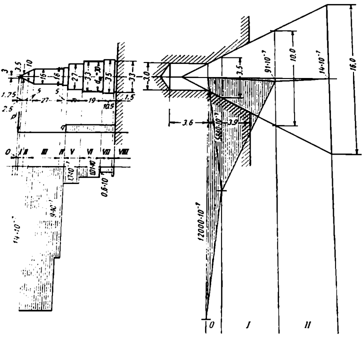
For a triangle with its apex pointing to the left (Fig. 15.11)

$$M_{a_k^1 x^0} = \Omega_\gamma \left(a_1 + \frac{2}{3} b \right) \tag{15.39}$$

$$M_{c_0^2 x^0} = \Omega_\gamma \left(a_1^2 + \frac{4 a_1 b}{3} + \frac{b^2}{2} \right) \tag{15.40}$$

$$M_{a_k^1 x^1} = \Omega_\gamma \left(\frac{a_1 b}{3} + \frac{b^2}{6} + a_1 a_2 + \frac{2}{3} a_2 b \right) \tag{15.41}$$

$$M_{c_0^2 x^1} = \Omega_\gamma \left(\frac{a_1^2 b}{3} + \frac{a_1 b^2}{3} - \frac{b^3}{10} + a_1^2 a_2 + \frac{4}{3} a_1 a_2 b + \frac{a_2 b^2}{2} \right) \tag{15.42}$$



(a) (b)

Fig. 15.8

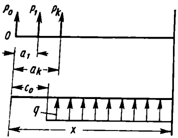


Fig. 15.9

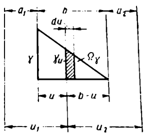


Fig. 15.10

For the loading diagram shown in Fig. 15.12, when $P = 5.64$ kgf and $q = 2.1$ kgf/cm, we obtain for the fixed end the following expressions

$$0 = \Theta_0 P M_{a^1_k x^0} - \frac{q}{2} M_{c^2_0 x^0}$$

$$0 = v_0 + \Theta_0 l - P M_{a^1_k x^1} - \frac{q}{2} M_{c^2_0 x^1}$$

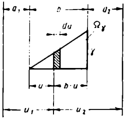


Fig. 15.11

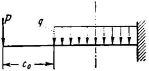


Fig. 15.12

Substitution of the numerical values gives the rotation angle and the deflection at the free end

$$\Theta_0 = 5.64 \cdot 663.0545 \cdot 10^{-7} + \frac{2.1}{2} \cdot 669.954 \cdot 10^{-7} = 4443 \cdot 10^{-7}$$

$$v_0 = 4443 \cdot 10^{-7} \cdot 8.825 + 5.64 \cdot 5148.73 \cdot 10^{-7} +$$

$$+ \frac{2.1}{2} \cdot 3705.19 \cdot 10^{-7} = -0.00063 \text{ cm}$$

Problem 2. Design of a hollow flour-grinding shaft.

Rigidity design of a hollow flour-grinding shaft is an example of application of biaxial moments of a flexibility diagram area.

The central part of such shafts is a hollow cast-iron cylindrical bushing pressed on steel axles rotating in bearings. The shafts have several separate parts of different rigidity along their length, as shown in Fig. 15.13 together with the flexibility diagram.

The uniformly distributed working load and the reactions of supports are shown in the q diagram. The angular and linear deflection formulas, when applied to loads of this type, have the following form

$$\Theta_x = \Theta_0 + \sum \frac{q}{2} M_{a^2 x^0} \tag{15.43}$$

$$v_x = v_0 + \Theta_0 x + \sum \frac{q}{2} M_{a^2 x^1} \tag{15.44}$$

where $M_{a^2 x^0}$ and $M_{a^2 x^1}$ are biaxial moments of the flexibility diagram, both moments being determined with respect to two parallel axes, one of which coincides with the beginning of the section from where the continuous load acts, while the other axis passes through section X, in which the displacement is determined. Moments in (15.43) and (15.44) are of the same order with respect to the first

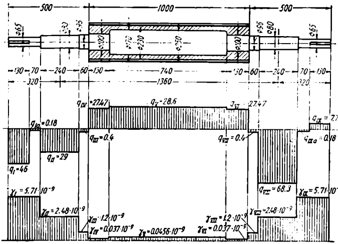

500
1000
500
⌀65
⌀90
⌀95
⌀100
⌀270
⌀230
⌀250
⌀100
⌀95
⌀80
⌀65
130
70
240
60
130
740
130
60
240
70
130
320
1360
320
$q_{IV} = 27.47$
$q_V = 28.6$
$q_{VI} = 27.47$
$q_{Ia} = 0.18$
$q_{IX} = 7.7$
$q_{III} = 0.4$
$q_{VII} = 0.4$
$q_{IXa} = 0.18$
$q_{II} = 29$
$q_I = 46$
$q_{VIII} = 68.3$
$\gamma_I = 5.71 \cdot 10^{-9}$
$\gamma_{IX} = 5.71 \cdot 10^{-9}$
$\gamma_{II} = 2.48 \cdot 10^{-9}$
$\gamma_{VIII} = 2.48 \cdot 10^{-9}$
$\gamma_{III} = 1.2 \cdot 10^{-9}$
$\gamma_{VII} = 1.2 \cdot 10^{-9}$
$\gamma_{IV} = 0.037 \cdot 10^{-9}$
$\gamma_V = 0.0456 \cdot 10^{-9}$
$\gamma_{VI} = 0.037 \cdot 10^{-9}$

Fig. 15.13

axis, while the order of moments with respect to section X in the second formula is a unit higher than in the first formula

$$M_{a^2x^0} = \int_{a_q}^{x} (u-a_q)^2 (x-u)^0 \, d\Omega_\gamma \tag{15.45}$$

$$M_{a^2x^1} = \int_{a_q}^{x} (u-a_q)^2 (x-u) \, d\Omega_\gamma \tag{15.46}$$

where $d\Omega_\gamma$ is an element of the flexibility diagram area; after integration we obtain

$$M_{a^2x^0} = \Omega_\gamma \left(\frac{b^2}{3} + a_1 b + a_1^2\right) \tag{15.47}$$

$$M_{a^2x^1} = \Omega_\gamma \left(\frac{b^3}{12} + \frac{a_1 b^2}{3} + \frac{a_1^2 b}{2} + \frac{a_2 b^2}{2} + a_1 a_2 b + a_1^2 a_2\right) \tag{15.48}$$

All the designations are shown in Fig. 15.14.

When calculating the results in practice, it is convenient to construct tables in which numerical values determined by polynomials are grouped in a column for each separate value of the transverse flexibility γ (Table 15.1).

Since the continuous loads of intensity q do not reach section X (if we move along the shaft from the origin of coordinates), the distributed loads of corresponding intensity are added and subtracted as usual to make the load q reach section X. For the shaft in Fig. 15.13 nine separate parts appeared on the flexibility diagram and eleven uniformly distributed loads.

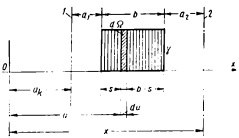


Fig. 15.14

All values were calculated by means of Table 15.1; for example, if $q_1 = 46$ kgf/cm we obtain

$$M_{a^2x1} = -2345 \cdot 10^4 \qquad (15.49)$$

After all the other distributed loads q were treated in the same way, the total moment was determined as

$$\sum \frac{q}{2} M_{a^2x1} = -2479 \cdot 10^{-4} \qquad (15.50)$$

and the following equations were composed for the supports

$$0 = v_0 + 168\Theta_0 - 0.24795; \quad 0 = v_0 + 3\Theta_0 - 0.00768$$

which give the values of the initial parameters of the flexural curve

$$\Theta_0 = 0.001767 \quad \text{and} \quad v_0 = -0.04888$$

and the linear displacements were calculated for all the check points.

* * *

Besides the method described above, there are several other methods of designing beams of variable cross sections; these methods are based on the moment-operational method presented in the following sections.

Table 15.1

Table for Calculation of Biaxial Moments. Values $a_1/u = \frac{b^3}{12} + \frac{a_1 b^2}{3} + \frac{a_1^2 b}{2} + \frac{a_2 b^2}{3} + a_1 a_2 b + a_1^2 a_2$ Relative to Point B

Flexibility	γ_I	γ_{II}	γ_{III}	γ_{IV}	γ_V	γ_{VI}	γ_{VII}	γ_{VIII}
$\Omega_\gamma \cdot 10^6$	$20 \cdot 5.71 = 114.2$ $7 \cdot 5.71 = 39.97$	$24 \cdot 2.48 = 59.52$	$6 \cdot 1.2 = 7.2$	$13 \cdot 0.0380 = 0.5$	$74 \cdot 0.0456 = 3.37$	$13 \cdot 0.0386 = 0.5$	$6 \cdot 1.2 = 7.2$	$12 \cdot 2.48 = 29.70$
b	28/7	24	6	13	74	13	6	12
a_2	148	124	118	105	31	18	12	0
q_I	0/20400	20/142,720	44/267,370	50/355,914	68/619,764	137/500,814	150/35,062	156/156,648
p_I	0/2446	7/53,800	31/140,035	37/211,331	50/466,820	124/413,910	137/29,305	143/129,702
a_{1a}	0/2446	7/53,800	31/140,035	37/211,331	50/466,820	124/413,910	137/293,205	143/129,702
p_{1a}		0/24,960	24/88,410	30/149,088	43/393,218	117/370,546	130/64,582	136/117,648
q_{II}		0/24,960	24/88,410	30/149,088	43/393,218	117/370,516	130/264,582	136/117,648
p_{II}			0/1434	6/18,640	19/193,167	93/240,099	106/177,606	112/80,784
q_{III}			0/1434	6/18,640	19/193,167	93/240,099	106/177,606	112/80,784
p_{III}				0/6098	13/155,897	87/211,897	100/158,562	106/72,648
q_{IV}				0/6098	13/155,897	87/211,897	100/158,562	106/72,648
p_{IV}					0/90,354	74/156,844	87/121,005	93/56,502
q_V					0/90,354	74/156,844	87/121,005	93/56,502
p_V						0/1197	13/3789	19/3222
q_{VI}						0/1197	13/3789	19/3222
p_{VI}							0/162	6/648
q_{VII}							0/162	6/648
p_{VII}								0/144
q_{VIII}								0/144

If the transverse flexibility of a beam (i.e., the inverse of rigidity) may be expressed by an algebraic polynomial in terms of the distance along the axis, then it is especially easy to design a beam. However, if the rigidity itself is expressed by a polynomial, say, $EI = g_0 + g_1x + g_2x^2 + \ldots$, then the required expression for the flexibility may be derived from

$$\frac{1}{EI} = \gamma_0 + \gamma_1 x + \gamma_2 x^2 + \ldots = \frac{1}{g_0 + g_1 x + g_2 x^2 + \ldots}$$

The flexibility polynomial may be also obtained from the given rigidity expression by approximation when several terms are taken into consideration.

The end reinforcement of beams often practised in engineering may be taken into account by approximating the ends with a parabolic curve (or a combination of curves) and by replacing the entire beam with a beam of a continuously varying cross section. The beam may be also divided into several beams of continuously varying cross sections, but a more complicated calculation method is necessary to solve this problem*.

Solutions obtained for beams of continuously varying cross sections are proved (in Chap. V) to be applicable in designing multispan beams of continuously varying sections within separate spans; these calculations become much simpler when the beam is symmetrical and especially if the loads are symmetrical or asymmetrical.

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* For example, this may be achieved by a combination of the moment method and functional discontinuators or pulse functions.

PART II

Moment-Operational Method: Theory and Application

Chapter III

Moment-Operational Method

16. PRINCIPLES OF MOMENT-OPERATIONAL METHOD

The first part of this book deals with the principles of the theory of higher moments and with the application of the moment method to various engineering problems. Two entirely different ways of development and application of higher moments were described; the geometrical way and the static-kinematic way.

In the first case, the topic discussed was the application of higher moments to the "geometry" of cross sections of bars subjected to bending or twisting, or any other force. Higher moments would inevitably arise and had to be used whenever a comparatively complicated or precise problem was to be solved. Higher moments were used as a subsidiary tool in these problems, the same way as moments of comparatively lower order, such as moments of zero order (areas), moments of the first order (static moments) and moments of the second order (moments of inertia), were used in simple problems on flexure (or on torsion of a circular cylinder shaft).

In the second case the moments of higher order were used in the form of external force moments for obtaining a series of values necessary for evaluating the strength and rigidity of beams, i.e., in static-kinematic beam evaluation. This trend of development and application of higher moments is a natural extension of the well-known concepts of the transverse force and the bending moment, or of linear and angular displacements taken as the first and zero moments, respectively, of the conjugate load of a beam presented by its flexibility diagram; the distributed load may be considered a moment of the "minus first" order. These ideas were illustrated by calculating the rigidity of a beam of a constant cross section.

In contrast to the first part of this book, the second part contains formulas derived without including the load in the calculations;

solutions are obtained by integral and differential operations with moment symbols, or, more accurately, by operations with composite functions representing moments of higher order of vectors of higher moments (the so-called bimoments).

To carry out successive computations and transforms several subsidiary formulas are given at the beginning for integration and differentiation of the indicated functions; these formulas are widely used in the further calculations.

As a result, a limited number of exact solutions was obtained and a wide range of problems was covered by means of the iteration method (similar to the iteration used in Pikar's method of the approximate solution of differential equations) to derive polynomial solutions of differential equations of bending.

The main quality of the moment-operational method consists of the fact that solutions obtained (approximate, although sufficiently precise for practical purposes) refer to any possible type of loading, both static and kinematic (the latter allows for pointed and distributed breaks and steps in the flexural curve).

The method is proved to be effective in solving certain problems which have been previously solved by other methods; in addition, a series of new solutions is presented for evaluating strength and rigidity.

The moment-operational method may be used in solving stability problems of rods subjected to kinematic loads.

Any problem solved by the moment-operational method obtains a general solution in terms of functions of higher moments; these functions are replaced by appropriate values when a particular load is considered. The solution does not become more complicated with any possible compound load.

The application of basic and subsidiary formulas to evaluate beams subjected to combined compression and flexure or combined tension and flexure in designing beams resting on elastic foundations and in some other problems concerning flexure of bars shows the possibility of solving any problem irrespective of the type of loading. If a certain problem is solved in terms of higher moments, then the symbols of the moments for any given load will be deciphered according to general rules.

Expressions for higher moments are derived in the form of final designing formulas only after the general solution was obtained. That is why all the intermediate analytical calculations are performed without considering any concrete loading and the necessary evaluating formulas are obtained as the results of mathematical operations over the symbols of moments.

These observations make it possible to consider the theory of higher moments as a kind of moment-operational calculus when a problem is solved by means of the moment-operational method.

The only thing necessary for application of the moment-operational method is to express the transverse flexibility, the restoration factor and other quantities in terms of the abscissa.

17. BIMOMENTS

When evaluating the rigidity of ordinary beams the angle Θ or deflection v is determined by integrating the curvature expression $k = \frac{M_U}{EI_U}$ once or twice, respectively; here $\frac{1}{EI_U} = \gamma_U$ is the flexibility at section U.

If the beam has a continuously changing rigidity, then γ_U is a function of the abscissa, as well as the bending moment M, through which the curvature k of the beam is expressed.

Denoting $\gamma_U = f(x)$ and $M = \varphi(x)$, we obtain $k = f(x)\,\varphi(x)$ and therefore the rotation angle Θ at section U is

$$\Theta_U = \Theta_0 + \int f(x)\,\varphi(x)\,dx \tag{17.1}$$

and the deflection at the same section is

$$v_U = v_0 + \Theta_0 u + \int\int f(x)\,\varphi(x)\,dx^2 \tag{17.2}$$

According to I. Bubnov, the transverse flexibility along the beam may be expressed by an algebraic polynomial, of the form

$$\gamma_U = \gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2!} + \gamma_3 \frac{x^3}{3!} + \gamma_4 \frac{x^4}{4!} + \dots \tag{17.3}$$

The bending moment is the first moment M_1 of the load which during the following multiple integration transforms in higher moments $\frac{M_2}{2!}$, $\frac{M_3}{3!}$, etc.

Thus, if the flexibility is expressed by an algebraic polynomial, then the curvature of the beam at section X is given by

$$\begin{aligned} k_X = \gamma_X M_1 &= \left[\gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2!} + \gamma_3 \frac{x^3}{3!} + \dots\right] M_1 = \\ &= \gamma_0 M_1 + \gamma_1 x M_1 + \gamma_2 \frac{x^2}{2!} M_1 + \dots \end{aligned} \tag{17.4}$$

When integrating this expression, we use products of the $\frac{x^m}{m!} \cdot \frac{M_n}{n!}$ type.

An expression of the following type

$$\mathfrak{M} = \frac{x^m}{m!} \cdot \frac{M_n}{n!} \tag{17.5}$$

is called *a bimoment of higher order*; it is actually a moment of order m of the moment of order n. Bimoments will be designated by the Gothic letter $\mathfrak{M}$.

If a moment $\frac{M_n}{n!}$ is represented by a vector applied at section X and directed normally to the plane of the drawing, then, according to formula (17.5), a bimoment is formed by multiplying this vector by its arm X raised to power m, similar to multiplying a force vector P by its arm raised to the required power in determining an ordinary n-th moment.

The moment-operational method will be also used to solve more complicated problems.

18. DIFFERENTIAL AND INTEGRAL BIMOMENTS

For simplicity of calculations in deriving differential and integral bimoments, formulas will be composed for multiple derivatives and integrals of $\mathfrak{M}$ in the form of tabular quantities.

The first differential bimoment $\overline{\mathfrak{M}}_1$ of the main bimoment $\mathfrak{M} = \frac{x^m}{m!}\cdot\frac{M_n}{n!}$ is derived as the first derivative from the product of two factors which are functions of the same independent variable x, so that

$$\overline{\mathfrak{M}}_1 = \frac{x^{m-1}}{(m-1)!}\cdot\frac{M_n}{n!} + \frac{x^m}{m!}\cdot\frac{M_{n-1}}{(n-1)!} \qquad (18.1)$$

since it was previously established that

$$\frac{d}{dx}\left(\frac{M_n}{n!}\right) = \frac{M_{n-1}}{(n-1)!} \qquad (18.2)$$

The subscript of $\overline{\mathfrak{M}}$ gives the order of the differential bimoment.

Similarly, the second differential bimoment $\overline{\mathfrak{M}}_2$ is the second derivative of $\overline{\mathfrak{M}}$ obtained by differentiating (18.1), therefore

$$\begin{aligned}\overline{\mathfrak{M}}_2 &= \frac{x^{m-2}}{(m-2)!}\cdot\frac{M_n}{n!} + \frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n-1}}{(n-1)!} + \\ &+ \frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n-1}}{(n-1)!} + \frac{x^m}{m!}\cdot\frac{M_{n-2}}{(n-2)!} = \\ &= \frac{x^{m-2}}{(m-2)!}\cdot\frac{M_n}{n!} + 2\,\frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n-1}}{(n-1)!} + \frac{x^m}{m!}\cdot\frac{M_{n-2}}{(n-2)!} \qquad (18.3)\end{aligned}$$

The sum of the exponents of the two factors for the main bimoment $\overline{\mathfrak{M}}$ is $(m + n)$. In the expression for the first differential bimoment this sum is $(m + n - 1)$ after the first differentiation; it is $(m + n - 2)$ for the second differential bimoment, and so on, that corresponds to the rules of differentiating usual power functions.

Similarly, an expression for the third bimoment $\overline{\mathfrak{M}}_3$ is written as the third derivative of the main bimoment or the first derivative of $\overline{\mathfrak{M}}_2$

$$\overline{\mathfrak{M}}_3=\frac{x^{m-3}}{(m-3)!}\cdot\frac{M_n}{n!}+3\frac{x^{m-2}}{(m-2)!}\cdot\frac{M_{n-1}}{(n-1)!}+$$
$$+3\frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n-2}}{(n-2)!}+\frac{x^m}{m!}\cdot\frac{M_{n-3}}{(n-3)!} \qquad (18.4)$$

For the fourth differential bimoment

$$\overline{\mathfrak{M}}_4=\frac{x^{m-4}}{(m-4)!}\cdot\frac{M_n}{n!}+4\frac{x^{m-3}}{(m-3)!}\cdot\frac{M_{n-1}}{(n-1)!}+$$
$$+6\frac{x^{m-2}}{(m-2)!}\cdot\frac{M_{n-2}}{(n-2)!}+4\frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n-3}}{(n-3)!}+\frac{x^m}{m!}\cdot\frac{M_{n-4}}{(n-4)!} \qquad (18.5)$$

Thus we arrive at the following general expression for a differential bimoment of order k

$$\overline{\mathfrak{M}}_k=\frac{d^k}{dx^k}\left(\frac{x^m}{m!}\cdot\frac{M_n}{n!}\right)=\sum_{s=0}^{s=k}\frac{k!}{s!}\;\frac{x^{m-s}}{(m-s)!}\;\frac{M_{n+s-k}}{(n+s-k)!} \qquad (18.6)$$

Formulas (18.1)-(18.5) are obtained as particular cases of (18.6) after substituting $k = 1, 2, 3, 4$.

The number of terms in each expression for a differential bimoment is always greater by one than the order of the derivative taken from the main bimoment $\mathfrak{M}$.

Successive numerical coefficients in the expressions for differential bimoments are the usual binomial coefficients which are easily found from Pascal's triangle (Table 18.1).

Any figure of Pascal's triangle is equal to the sum of two figures: one in the preceding row and column and one in the preceding row of the same column.

The rows of the triangle contain numerical coefficients of the successive terms derived for the differential bimoments. For example, the fourth line contains figures 1, 4, 6, 4, 1, i.e., the ones obtained for the fourth differential bimoment $\overline{\mathfrak{M}}_4$, etc.

We switch now from the differential to the integral bimoments, which are more widely applied in rigidity computations of various beams.

The first integral bimoment $\mathfrak{M}_1$ is the first integral of the main bimoment $\mathfrak{M}=\frac{x^m}{m!}\cdot\frac{M_n}{n!}$, i.e.,

$$\mathfrak{M}_1=\int\frac{x^m}{m!}\cdot\frac{M_n}{n!}\,dx=\int\frac{x^m}{m!}\,d\left[\frac{M_{n+1}}{(n+1)!}\right]=$$
$$=\frac{x^m}{m!}\cdot\frac{M_{n+1}}{(n+1)!}-\int\frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n+1}}{(n+1)!}\,dx$$

Table 18.1

Pascal's Triangle for Coefficients of Differential and Integral Bimoments

No. of differential bimoment													
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1
13	1	13	78	286	715	1287	1716	1716	1287	715	286	78	13
No. of integral bimoment	1	2	3	4	5	6	7	8	9	10	11	12	13

Integration by parts according to the general formula

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

and presentation of $\frac{M_n}{n!}$ in terms of the next (higher) order

$$\frac{M_n}{n!} = \frac{d}{dx}\left[\frac{M_{n+1}}{(n+1)!}\right]$$

does not give the final expression, because partial integration may be continued by applying a similar procedure to the integrand expression

$$\frac{M_{n+1}}{(n+1)!} = \frac{d}{dx}\left[\frac{M_{n+2}}{(n+2)!}\right]$$

The final expression of the first integral bimoment is

$$\mathfrak{M}_1 = \frac{x^m}{m!} \cdot \frac{M_{n+1}}{(n+1)!} + \frac{x^{m-1}}{(m-1)!} \cdot \frac{M_{n+2}}{(n+2)!} + \\ + \frac{x^{m-2}}{(m-2)!} \cdot \frac{M_{n+3}}{(n+3)!} - \frac{x^{m-3}}{(m-3)!} \cdot \frac{M_{n+4}}{(n+4)!} + \dots \tag{18.7}$$

Structural regularities are thus revealed in polynomials obtained for integral bimoments, so that there is no need for further integration because the other terms may be similarly written.

The sum of the exponents of both factors in the expression for the basic bimoment is $(m + n)$. The sum of powers in each term of the polynomial expression derived for the first integral bimoment $\mathfrak{M}_1$ is $(m + n + 1)$ according to the rules of integration.

Let us now derive an expression for the second integral bimoment, which is the second integral of the basic bimoment $\mathfrak{M} = \frac{x^m}{m!} \cdot \frac{M_n}{n!}$ and the first integral of $\mathfrak{M}_1$. Using formula (18.7) for the construction of a new integral formed as previously according to the principle of (7.5), we establish the differential relationship between two successive moments.

The second moment is determined by

$$\mathfrak{M}_2 = \int\int \frac{x^m}{m!} \cdot \frac{M_n}{n!}\, dx^2$$

A polynomial derived by integrating each term in expression (18.7) is rearranged according to the orders of higher moments, the following formula being a result of integration, its first line contains terms derived from integration of the first term of (18.7), the second line refers to the second term, etc. Therefore we obtain

$$\mathfrak{M}_2 = \frac{x^m}{m!} \cdot \frac{M_{n+2}}{(n+2)!} - 2\frac{x^{m-1}}{(m-1)!} \cdot \frac{M_{n+3}}{(n+3)!} + 3\frac{x^{m-2}}{(m-2)!} \times \\ \times \frac{M_{n+4}}{(n+4)!} - 4\frac{x^{m-3}}{(m-3)!} \cdot \frac{M_{n+5}}{(n+5)!} + \dots \tag{18.8}$$

Successive integral bimoments are obtained likewise. Numerical coefficients in the successive terms of the alternating polynomials resulting from integration, may be taken from Pascal's triangle by writing out figures from the column.

For example, the fifth column of the triangle may be used to obtain numerical coefficients 1, 5, 15, 35, 70, 126... for succesive terms of the fifth integral bimoment arranged according to the rules outlined above

$$\begin{aligned}\mathfrak{M}_5 = \frac{x^m}{m!}\cdot\frac{M_{n+5}}{(n+5)!} - 5\frac{x^{m-1}}{(m-1)!}\cdot\frac{M_{n+6}}{(n+6)!} + \\ + 15\frac{x^{m-2}}{(m-2)!}\cdot\frac{M_{n+7}}{(n+7)!} - 35\frac{x^{m-3}}{(m-3)!}\cdot\frac{M_{n+8}}{(n+8)!} + \\ 70\frac{x^{m-4}}{(m-4)!}\cdot\frac{M_{n+9}}{(n+9)!} - 126\frac{x^{m-5}}{(m-5)!}\cdot\frac{M_{n+10}}{(n+10)!} + \dots \end{aligned} \qquad (18.9)$$

Formulas for integral bimoments may be either exact or approximate depending on the power m and the number of terms in the series used for calculation; the more terms are included, the more precise is the solution given by the approximate formula.

The following conclusions relating to the structure of formulas may be drawn from comparison of formulas obtained for integral and differential bimoments.

All the terms of differential bimoments have the same sign; the first term of all the polynomials always contains a factor $\frac{M_n}{n}$ while the power of the geometrical factor x is equal to m minus the order of the derivative used in obtaining the bimoment. The order of moment gradually decreases in the successive terms of the series as required by differentiation rules, whereas the exponents of x gradually increases according to integration rules; in the last term the power of x is always equal to m. The sum of powers of both factors in each term of every subsequent series is one unit lower than that of the preceding series. In the basic bimoment $\mathfrak{M} = \frac{x^m}{m!}\cdot\frac{M_n}{n!}$ this sum is $(m+n)$: in the first bimoment it is $(m+n-1)$; in the second bimoment, $(m+n-2)$, etc.

All the terms of integral bimoments have alternating signs; the first term of all the polynomials contains a factor $\frac{x^m}{m}$, while the order of moment M in the first term is equal to n plus the order of the integral of the main bimoment $\mathfrak{M}$ used in obtaining the integral bimoment. The order of moment gradually increases in the successive terms of the series as required by integration rules, whereas the power of x gradually decreases according to differentiation rules.

The sum of exponents of both factors in each term of every subsequent integral bimoment is one unit higher than that of the preceding bimoment, according to integration rules. In the first integral bimoment it is equal to $(m + n + 1)$; in the second, $(m + n + 2)$; in the third, $(m + n + 3)$, etc.

All the expressions obtained may be generalized in the following formulas for integral bimoments:

(a) The first integral bimoment

$$\mathfrak{M}_1 = \sum_{k=0}^{k=m} (-1)^k \frac{x^{m-k}}{(m-k)!} \cdot \frac{M_{n+k+1}}{(n+k+1)!} \tag{18.10}$$

b) The second integral bimoment

$$\mathfrak{M}_2 = \sum_{k=0}^{k=m} (-1)^k (k+1) \frac{x^{m-k}}{(m-k)!} \cdot \frac{M_{n+k+2}}{(n+k+2)!} \tag{18.11}$$

c) The third integral bimoment

$$\mathfrak{M}_3 = \sum_{k=0}^{k=m} (-1)^k \frac{(k+1)(k+2)}{1\cdot 2} \cdot \frac{x^{m-k}}{(m-k)!} \frac{M_{n+k+3}}{(n+k+3)!} \tag{18.12}$$

(d) The fourth integral bimoment

$$\mathfrak{M}_4 = \sum_{k=0}^{k=m} (-1)^k \frac{(k+1)(k+2)(k+3)}{1\cdot 2\cdot 3} \cdot \frac{x^{m-k}}{(m-k)!} \frac{M_{n+k+4}}{(n+k+4)!} \tag{18.13}$$

(e) The fifth integral bimoment

$$\mathfrak{M}_5 = \sum_{k=0}^{k=m} (-1)^k \frac{(k+4)!\, x^{m-k}}{4!k!\,(m-k)!} \frac{M_{n+k+5}}{(n+k+5)!} \tag{18.14}$$

and, finally, in the general form

(f) The p-th integral bimoment

$$\mathfrak{M}_p = \sum_{k=0}^{k=m} (-1)^k \frac{(k+p-1)!}{(p-1)!k!} \cdot \frac{x^{m-k}}{(m-k)!} \cdot \frac{M_{n+k+p}}{(n+k+p)!} \tag{18.15}$$

For direct applications in the following calculations the first four integral bimoments will be presented herein by assuming $m = 1$, 2, 3 and 4, successively.

The first integral bimoments

$$\int x \frac{M_n}{n!}\, dx = x \frac{M_{n+1}}{(n+1)!} - \frac{M_{n+2}}{(n+2)!} \tag{18.16}$$

$$\int \frac{x^2}{2} \cdot \frac{M_n}{n!}\, dx = \frac{x^2}{2!} \cdot \frac{M_{n+1}}{(n+1)!} - x \frac{M_{n+2}}{(n+2)!} + \frac{M_{n+3}}{(n+3)!} \tag{18.17}$$

$$\int \frac{x^3}{3!} \cdot \frac{M_n}{n!}\, dx = \frac{x^3 M_{n+1}}{3!\,(n+1)!} + \frac{x^2}{2} \cdot \frac{M_{n+2}}{(n+2)!} + x \frac{M_{n+3}}{(n+3)!} - \frac{M_{n+4}}{(n+4)!} \tag{18.18}$$

$$\int \frac{x^4}{4!}\cdot\frac{M_n}{n!}\,dx=\frac{x^4}{4!}\cdot\frac{M_{n+1}}{(n+1)!}-\frac{x^3}{3!}\cdot\frac{M_{n+2}}{(n+2)!}+\frac{x^2}{2!}\cdot\frac{M_{n+3}}{(n+3)!}-x\,\frac{M_{n+4}}{(n+4)!}+\frac{M_{n+5}}{(n+5)!} \tag{18.19}$$

The second integral bimoments

$$\iint x\,\frac{M_n}{n!}\,dx^2=x\,\frac{M_{n+2}}{(n+2)!}-2\,\frac{M_{n+3}}{(n+3)!} \tag{18.20}$$

$$\iint \frac{x^2}{2}\cdot\frac{M_n}{n!}\,dx^2=\frac{x^2}{2}\cdot\frac{M_{n+2}}{(n+2)!}-2x\,\frac{M_{n+3}}{(n+3)!}+3\,\frac{M_{n+4}}{(n+4)!} \tag{18.21}$$

$$\iint \frac{x^3}{3!}\cdot\frac{M_n}{n!}\,dx^2=\frac{x^3}{3!}\cdot\frac{M_{n+2}}{(n+2)!}-2\,\frac{x^2}{2}\cdot\frac{M_{n+3}}{(n+3)!}+3x\,\frac{M_{n+4}}{(n+4)!}-4\,\frac{M_{n+5}}{(n+5)!} \tag{18.22}$$

$$\iint \frac{x^4}{4}\cdot\frac{M_n}{n!}\,dx^2=\frac{x^4}{4!}\cdot\frac{M_{n+2}}{(n+2)!}-2\,\frac{x^3}{3!}\cdot\frac{M_{n+3}}{(n+3)!}+3\,\frac{x^2}{2}\cdot\frac{M_{n+4}}{(n+4)!}-4x\,\frac{M_{n+5}}{(n+5)!}+5\,\frac{M_{n+6}}{(n+6)!} \tag{18.23}$$

The third integral bimoments

$$\iiint x\,\frac{M_n}{n!}\,dx^3=x\,\frac{M_{n+3}}{(n+3)!}-3\,\frac{M_{n+4}}{(n+4)!} \tag{18.24}$$

$$\iiint \frac{x^2}{2}\cdot\frac{M_n}{n!}\,dx^3=\frac{x^2}{2}\cdot\frac{M_{n+3}}{(n+3)!}-3x\,\frac{M_{n+4}}{(n+4)!}+6\,\frac{M_{n+5}}{(n+5)!} \tag{18.25}$$

$$\iiint \frac{x^3}{3!}\cdot\frac{M_n}{n!}\,dx^3=\frac{x^3}{3!}\cdot\frac{M_{n+3}}{(n+3)!}-3\,\frac{x^2}{2!}\cdot\frac{M_{n+4}}{(n+4)!}+6x\,\frac{M_{n+5}}{(n+5)!}-10\,\frac{M_{n+6}}{(n+6)!} \tag{18.26}$$

$$\iiint \frac{x^4}{4!}\cdot\frac{M_n}{n!}\,dx^3=\frac{x^4}{4!}\cdot\frac{M_{n+3}}{(n+3)!}-3\,\frac{x^3}{3!}\cdot\frac{M_{n+4}}{(n+4)!}+6\,\frac{x^2}{2}\cdot\frac{M_{n+5}}{(n+5)!}-10x\,\frac{M_{n+6}}{(n+6)!}+15\,\frac{M_{n+7}}{(n+7)!} \tag{18.27}$$

The fourth integral bimoments

$$\iiiint x\,\frac{M_n}{n!}\,dx^4=x\,\frac{M_{n+4}}{(n+4)!}-4\,\frac{M_{n+5}}{(n+5)!} \tag{18.28}$$

$$\iiiint \frac{x^2}{2}\cdot\frac{M_n}{n!}\,dx^4=\frac{x^2}{2}\cdot\frac{M_{n+4}}{(n+4)!}-4x\,\frac{M_{n+5}}{(n+5)!}+10\,\frac{M_{n+6}}{(n+6)!} \tag{18.29}$$

$$\iiiint \frac{x^3}{3!}\cdot\frac{M_n}{n!}\,dx^4=\frac{x^3}{3!}\cdot\frac{M_{n+4}}{(n+4)!}-4\,\frac{x^2}{2}\cdot\frac{M_{n+5}}{(n+5)!}+10x\,\frac{M_{n+6}}{(n+6)!}-20\,\frac{M_{n+7}}{(n+7)!} \tag{18.30}$$

$$\iiiint \frac{x^4}{4!}\cdot\frac{M_n}{n!}\,dx^4=\frac{x^4}{4!}\cdot\frac{M_{n+4}}{(n+4)!}-4\,\frac{x^3}{3!}\cdot\frac{M_{n+5}}{(n+5)!}+10\,\frac{x^2}{2}\cdot\frac{M_{n+6}}{(n+6)!}-20x\,\frac{M_{n+7}}{(n+7)!}+35\,\frac{M_{n+8}}{(n+8)!},\ \text{etc.} \tag{18.31}$$

19. APPLICATION OF MOMENT-OPERATIONAL METHOD FOR SOLVING LINEAR DIFFERENTIAL EQUATIONS

Consider a linear differential equation of the n-th order

$$\frac{d^nV}{dx^n}+f_{n-1}(x)\frac{d^{n-1}V}{dx^{n-1}}+f_{n-2}(x)\frac{d^{n-2}V}{dx^{n-2}}+\dots+$$
$$+f_2(x)\frac{d^2V}{dx^2}+f_1(x)\frac{dV}{dx}+f_0(x)V=D \qquad (19.1)$$

where D is a moment of a certain order; for example

$$D=M_1;\quad D=M_0=Q;\quad D=M_1=q$$

An exact solution of this equation appears to be rather difficult to obtain even if special functions are used. If the equation is to be solved in terms of trigonometrical functions, then trigonometrical moments' techniques may be used as proposed by the author for solving beams subjected to combined transversal and longitudinal load. However, if an approximate solution is required in the form of an algebraic polynomial, then the method of successive approximations may be applied in terms of moment symbols of various orders similar to Pikar's method of solving general differential equations. The solution may be given by the Chebyshev polynomial representing the least-maximum-absoluto-error approximation; in this case it is highly advisable to solve the problem first for a single point load and then to extend this solution to any given load by means of moment symbolics. This procedure actually means loading the influence line with a combined load, the influence line expression being derived by the iterative method.

The general way of solving linear differential equation (19.1) by the iterative method consists of the following:

(1) to obtain the first approximation we assume that all the terms in the left-hand part of the equation, except the first one, are equal to zero;

(2) to determine the unknown function $V=V_1$ and all its derivatives we solve the equation

$$\frac{d^nV}{\partial x^n}=D \qquad (19.2)$$

and find all its derivatives

$$\frac{dV_1}{dx};\ \frac{d^2V_1}{dx^2}\dots\frac{d^{n-1}V_1}{dx^{n-1}}$$

(3) by substituting the values of V_1; $\frac{dV_1}{dx}$; $\frac{d^2V_1}{dx^2}$ etc., found for the first approximation, into the right-hand part of the following

equation

$$\frac{d^nV}{dx^n} = D - f_0(x)\,V_1 - f_1(x)\,\frac{dV_1}{dx^2} - f_2(x)\,\frac{d^2V_1}{dx^2} - \ldots -$$
$$- f_{n-2}(x)\,\frac{d^{n-2}V_1}{dx^{n-2}} - f_{n-1}(x)\,\frac{d^{n-1}V_1}{dx^{n-1}} \qquad (19.3)$$

we solve the latter for V to determine the second approximation, etc.

Thus, the infinite moment series obtained in moment symbols of various integer orders is nothing else but the expansion in Taylor-Maclaurin's series of the function which is the exact solution of the differential equation (19.1). If such an exact solution is not found then the obtained infinite series is in fact the Taylor-Maclaurin's expansion of the same but not yet found solution. The speed of convergence is judged by estimating the value of the first rejected term.

Equation (19.1) is especially easily solved if its coefficients are expressed by algebraic polynomials in terms of x.

A third order differential equation will now be used to illustrate the application of a method which may be used for any coefficients.

For $n = 3$ equation (19.1) gives

$$\frac{d^3V}{dx^3} = D - f_0(x)\,V - f_1(x)\,\frac{dV}{dx} - f_2(x)\,\frac{d^2V}{dx^2} \qquad (19.4)$$

Although the coefficients of this equation are variable, they may be considered constant within a sufficiently small interval of the argument x:

$$f_0(x) = A; \quad f_1(x) = B; \quad f_2(x) = C$$

First we obtain the solution of equation (19.4) in its zero approximation by neglecting all the terms except the first one in the right-hand side

$$\frac{d^3V}{dx^3} = D = M_0; \quad \frac{d^2V}{dx^2} = \frac{M_1}{1}; \quad \frac{dV}{dx} = \frac{M_2}{2!}; \quad V = \frac{M_3}{3!} \qquad (19.5)$$

Substitution of V and its derivatives into (19.4) and subsequent integration gives another term of the first approximation

$$\dddot{V} = -C\frac{M_1}{1} - B\frac{M_2}{2!} - A\frac{M_3}{3!}; \quad \ddot{V} = -C\frac{M_2}{2!} - B\frac{M_3}{3!} - A\frac{M_4}{4!}$$
$$\dot{V} = -C\frac{M_3}{3!} - B\frac{M_4}{4!} - A\frac{M_5}{5!}; \quad V = -C\frac{M_4}{4!} - B\frac{M_5}{5!} - A\frac{M_6}{6!}$$

Similarly, another term of the third approximation of the solution of equation (19.4) is obtained by

$$\dddot{V} = C^2\frac{M_2}{2!} + 2BC\frac{M_3}{3!} + (2AC + B^2)\frac{M_4}{4!} + 2AB\frac{M_5}{5!} + A^2\frac{M_6}{6!}$$
$$\ddot{V} = C^2\frac{M_3}{3!} + 2BC\frac{M_4}{4!} + (2AC + B^2)\frac{M_5}{5!} + 2AB\frac{M_6}{6!} + A^2\frac{M_7}{7!}$$

$$\dot{V} = C^2\frac{M_4}{4!} + 2BC\frac{M_5}{5!} + (2AC + B^2)\frac{M_6}{6!} + 2AB\frac{M_7}{7!} + A^2\frac{M_8}{8!}$$

$$V = C^2\frac{M_5}{5!} + 2BC\frac{M_6}{6!} + (2AC + B^2)\frac{M_7}{7!} + 2AB\frac{M_8}{8!} + A^2\frac{M_9}{9!}$$

Finally, the solution of equation (19.4) is given in its second approximation by

$$\dddot{V} = M_0 - C\frac{M_1}{1} - (B - C^2)\frac{M_2}{2!} - (A - 2BC)\frac{M_3}{3!} + (2AC + B^2)\frac{M_4}{4!} + 2AB\frac{M_5}{5!} + A^2\frac{M_6}{6!} \tag{19.6}$$

$$\ddot{V} = \frac{M_1}{1} - C\frac{M_2}{2!} - (B - C^2)\frac{M_3}{3!} - (A - 2BC)\frac{M_4}{4!} + (2AC + B^2)\frac{M_5}{5!} + 2AB\frac{M_6}{6!} + A^2\frac{M_7}{7!} \tag{19.7}$$

$$\dot{V} = \frac{M_2}{2!} - C\frac{M_3}{3!} - (B - C^2)\frac{M_4}{4!} - (A - 2BC)\frac{M_5}{5!} + (2AC + B^2)\frac{M_6}{6!} + 2AB\frac{M_7}{7!} + A^2\frac{M_8}{8!} \tag{19.8}$$

$$V = \frac{M_3}{3!} - C\frac{M_4}{4!} - (B - C^2)\frac{M_5}{5!} - (A - 2BC)\frac{M_6}{6!} + (2AC + B^2)\frac{M_7}{7!} + 2AB\frac{M_8}{8!} + A^2\frac{M_9}{9!} \tag{19.9}$$

Chapter IV
Rigidity of Non-Uniform Beams

20. DETERMINATION OF DISPLACEMENTS BY COEFFICIENTS OF FLEXIBILITY POLYNOMIAL EXPRESSION

Three different methods of determining displacements in non-uniform beams, as well as designing non-uniform continuous beams, will be examined herein under the usual assumption that

$$k \approx \frac{d^2v}{dx^2}$$

The first method described in the present section can be only applied when the transverse flexibility $\gamma = \frac{1}{EI}$, which is reverse to rigidity, is expressed by an algebraic polynomial

$$\gamma_X = \gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2} + \gamma_3 \frac{x^3}{3!} + \gamma_4 \frac{x^4}{4!} + \dots \qquad (20.1)$$

This gives the following expression for the curvature of a beam at section X with abscissa x

$$k_X = \frac{d^2v}{dx^2} = \frac{M_1}{EI} = \gamma_X M_1 = \gamma_0 M_1 + \gamma_1 x M_1 + \\ + \gamma_2 \frac{x^2}{2!} M_1 + \gamma_3 \frac{x^3}{3!} M_1 + \gamma_4 \frac{x^4}{4!} M_1 + \dots \qquad (20.2)$$

in which the first moment (M_1) is the bending or static moment of all the forces, both active and reactive, applied to the left of the examined section.

Integrating formula (20.2) once, twice, three and four times and using subsidiary formulas (18.17)-(18.23) and (18.28)-(18.31), we obtain expressions first for the angular displacements and then for the linear deflections of the areas outlined by the left-hand part of the flexural curve, and for static moments of these areas.

The first integral of (20.2) and formulas (18.16)-(18.19) give an expression for the rotation angle of the flexural curve at an arbitrary section X with abscissa x*:

$$\begin{aligned}\Theta_X = \Theta_0 + \gamma_0 \frac{M_2}{2!} + \gamma_1 \left(x \frac{M_2}{2!} - \frac{M_3}{3!} \right) + \\ + \gamma_2 \left(\frac{x^2}{2!} \cdot \frac{M_2}{2!} - x \frac{M_3}{3!} + \frac{M_4}{4!} \right) + \\ + \gamma_3 \left(\frac{x^3}{3!} \cdot \frac{M_2}{2!} - \frac{x^2}{2!} \cdot \frac{M_3}{3!} + x \frac{M_4}{4!} - \frac{M_5}{5!} \right) + \ldots \quad (20.3)\end{aligned}$$

Deflection of the beam in the same section is expressed by a formula derived after (20.2) was integrated twice [which is more easily done than integration of (20.3)] with formulas (18.20)-(18.23) taken into account

$$\begin{aligned}v_X = v_0 + \Theta x + \gamma_0 \frac{M_3}{3!} + \gamma_1 \left(x \frac{M_3}{3!} - 2 \frac{M_4}{4!} \right) + \\ + \gamma_2 \left(\frac{x^2}{2!} \cdot \frac{M_3}{3!} - 2x \frac{M_4}{4!} + 3 \frac{M_5}{5!} \right) + \\ + \gamma_3 \left(\frac{x^3}{3!} \cdot \frac{M_3}{3!} - 2 \frac{x^2}{2!} \cdot \frac{M_4}{4!} + 3x \frac{M_5}{5!} - 4 \frac{M_6}{6!} \right) + \ldots \quad (20.4)\end{aligned}$$

Similarly, the following formula for the left-hand part of the area outlined by the flexural curve (in the OX interval) is derived by integrating (20.2) three times with (18.24)-(18.27) taken into account

$$\begin{aligned}\Omega_X = v_0 x + \Theta_0 \frac{x^2}{2!} + \gamma_0 \frac{M_4}{4!} + \gamma_1 \left(x \frac{M_4}{4!} - 3 \frac{M_5}{5!} \right) + \\ + \gamma_2 \left(\frac{x^2}{2!} \cdot \frac{M_4}{4!} - 3x \frac{M_5}{5!} + \frac{M_6}{6!} \right) + \\ + \gamma_3 \left(\frac{x^3}{3!} - 3 \frac{x^2}{2!} \cdot \frac{M_5}{5!} + 6x \frac{M_6}{6!} - 10 \frac{M_7}{7!} \right) + \ldots \quad (20.5)\end{aligned}$$

Finally, the fourth integral** of (20.2) and subsidiary formulas of Sec. 18 give the following expression for the static moment of the area mentioned previously with respect to point X

---

* Here and further Θ_0 and v_0 are the initial rotation angle and deflection, respectively, at the origin of the coordinate system, which generally coincides with the left end of the beam.

** Further integration gives formulas for the second moment (divided by 2!), the third moment (divided by 3!), the fourth moment (divided by 4!) of the same area Ω_X.

We shall use only the four formulas derived herein; the first two formulas relate to rigidity design directly, the other two formulas are used in calculating Mohr's displacement integral for distributed loads of rectangular and triangular shapes.

$$S_X = v_0 \frac{x^2}{2} + \Theta_0 \frac{x^3}{3!} + \gamma_0 \frac{M_5}{5!} + \gamma_1 \left(x \frac{M_5}{5!} - 4 \frac{M_6}{6!} \right) + \\ + \gamma_2 \left(\frac{x^2}{2!} \cdot \frac{M_5}{5!} - 4x \frac{M_6}{6!} + 10 \frac{M_7}{7!} \right) + \\ + \gamma_3 \left(\frac{x^3}{3!} \cdot \frac{M_5}{5!} - 4 \frac{x^2}{2!} \cdot \frac{M_6}{6!} + 10x \frac{M_7}{7!} - 20 \frac{M_8}{8!} \right) + \dots \qquad (20.6)$$

Higher moments appearing in (20.3)-(20.6) must be calculated beforehand according to the formulas presented in Chap. I.

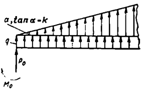

Fig. 20.1

Consider the particular case shown in Fig. 20.1: a point force couple $\overline{M}$ and a point load P are applied to the section with abscissa $x = 0$, i.e., at the origin of coordinates, while the simplest distributed loads, such as the uniformly distributed and the linearly varying loads, act upon the beam along the entire abscissa axis x; thus, according to formula (I) of Sec. 6

$$\frac{M_n}{n!} = \overline{M}_0 \frac{x^{n-1}}{(n-1)!} + P_0 \frac{x^n}{n!} + q \frac{x^{n+1}}{(n+1)!} + k \frac{x^{n+2}}{(n+2)!} \qquad (20.7)$$

By substituting the values of $n = 2, 3, 4 \dots$ into formula (20.7) we find the values of higher moments for substitution into formulas (20.3)-(20.6), which result in the following formulas

$$\Theta_X = \Theta_0 + \overline{M}_0 \left(\gamma_0 x + \gamma_1 \frac{x^2}{2!} + \gamma_2 \frac{x^3}{3!} + \gamma_3 \frac{x^4}{4!} + \dots \right) + \\ + P_0 \left(\gamma_0 \frac{x^2}{2!} + \gamma_1 \frac{2x^3}{3!} + \gamma_2 \frac{3x^4}{4!} + \gamma_3 \frac{4x^5}{5!} + \dots \right) + \\ + q \left(\gamma_0 \frac{x^3}{3!} + \gamma_1 \frac{3x^4}{4!} + \gamma_2 \frac{6x^5}{5!} + \gamma_3 \frac{10x^6}{6!} + \dots \right) + \\ + k \left(\gamma_0 \frac{x^4}{4!} + \gamma_1 \frac{4x^5}{5!} + \gamma_2 \frac{10x^6}{6!} + \gamma_3 \frac{20x^7}{7!} + \dots \right) \qquad (20.8)$$

$$v_X = v_0 + \Theta_0 x + \overline{M}_0 \left(\gamma_0 \frac{x^2}{2!} + \gamma_1 \frac{x^3}{3!} + \gamma_2 \frac{x^4}{4!} + \gamma_3 \frac{x^5}{5!} + \dots \right) + \\ + P_0 \left(\gamma_0 \frac{x^3}{3!} + \gamma_1 \frac{2x^4}{4!} + \gamma_2 \frac{3x^5}{5!} + \gamma_3 \frac{4x^6}{6!} + \dots \right) + \\ + q \left(\gamma_0 \frac{x^4}{4!} + \gamma_1 \frac{3x^5}{5!} + \gamma_2 \frac{6x^6}{6!} + \gamma_3 \frac{10x^7}{7!} + \dots \right) + \dots \qquad (20.9)$$

$$\Omega_X = v_0 x + \Theta_0 \frac{x^2}{2!} + \overline{M}_0 \left(\gamma_0 \frac{x^3}{3!} + \gamma_1 \frac{x^4}{4!} + \gamma_2 \frac{x^5}{5!} + \gamma_3 \frac{x^6}{6!} + \dots \right) + \\ + P_0 \left(\gamma_0 \frac{x^4}{4!} + \gamma_1 \frac{2x^5}{5!} + \gamma_2 \frac{3x^6}{6!} + \gamma_3 \frac{4x^7}{7!} + \dots \right) + \dots \qquad (20.10)$$

$$S_x = v_0 \frac{x^2}{2!} + \Theta_0 \frac{x^3}{3!} + \overline{M}_0 \left(\gamma_0 \frac{x^4}{4!} + \gamma_1 \frac{x^5}{5!} + \gamma_2 \frac{x^6}{6!} + \right.$$
$$\left. + \gamma_3 \frac{x^7}{7!} + \ldots \right) + P_0 \left(\gamma_0 \frac{x^5}{5!} + \gamma_1 \frac{2x^6}{6!} + \gamma_2 \frac{3x^7}{7!} + \gamma_3 \frac{4x^8}{8!} + \ldots \right) + \ldots \tag{20.11}$$

For the particular case under consideration N. Snitko obtained formulas for angular and linear deflections; he also gave the equation of the flexural curve for a non-uniform beam by expressing the flexibility by a trinomial taking into account the loads contained in formula (20.7). Thus, the solution given by N. Snitko refers only to the initial conditions. The moment-operational method used in the present section gives the general solution for any loads including "kinematic" ones, which allow for point and distributed steps and special points in the flexural curve. Snitko's solution is based on the direct integration of the flexural curve of differential equation; the flexibility expression must be derived every time a force or any other loading appears on the beam, and the reference point must be transferred to this point according to the initial parameters procedure, these calculations require a large amount of labour. All these considerations and the results obtained further prove the efficiency of the moment-operational method in calculating continuous non-uniform beams.

If the flexibility variations of a beam follow any given law presented in a graphical or analytical form, then it may be approximated with sufficient accuracy by a polynomial with a comparatively small number of terms.

Two other procedures of designing non-uniform beams are outlined herein, which, unlike the method mentioned above, do not require the flexibility to be expressed by an algebraic polynomial and can be applied to continuously non-uniform beams with rigidity polynomial expressed by various formulas.

$$EI = g_0 + g_1 x + g_2 \frac{x^2}{2!} + \ldots$$

may be solved after the flexibility expression was derived either by direct division

$$\frac{1}{EI} = \gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2!} + \ldots = \frac{1}{g_0 + g_1 x + g^2 \frac{x^2}{2!} + \ldots}$$

or by approximation.

Brackets used in engineering may be taken account of through approximating by parabolic curves and introducing a continuously non-uniform beam or a combined beam in place of the given beam, the combined beam being considered as a succession of beams of

Solutions obtained for a continuously non-uniform beam may be applied to the design of non-uniform continuous beams; the calculations may be substantially simplified when the beam is symmetrical and especially when it is loaded symmetrically or asymmetrically.

For example, examine a simple beam freely supported at its ends, with its rigidity varying continuously and symmetrically with respect to the middle (the maximum being at the centre), as shown in Fig. 20.2.

Let the flexibility of section X with abscissa x from the origin of coordinates be expressed by

$$\gamma_X = \frac{1}{EI_X} = \frac{1}{EI_C} \cdot \frac{EI_C}{EI_X} = \frac{1}{EI_C}\, i \tag{20.12}$$

where

$$i = 1 - (1 - n)\frac{(l - 2x)^2}{l^2} \tag{20.13}$$

This expression may be transformed into

$$i = n + (1 - n)\frac{4}{l}x - (1 - n)\frac{4}{l^2}x^2 \tag{20.14}$$

which results in the following polynomial

$$\gamma = \gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2}$$

where

$$\gamma_0 = \frac{n}{EI_C}\,; \quad \gamma_1 = \frac{1-n}{EI_C} \cdot \frac{4}{l}$$

$$\gamma_2 = -(1 - n)\frac{8}{l^2 EI_C} \tag{20.15}$$

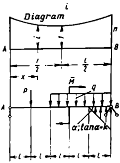

Fig. 20.2

Suppose the beam is subjected to flexure under the influence of a composite load shown in Fig. 20.2.

Formulas (20.3) and (20.4) may be used to derive rotation angles and deflections

$$\Theta = \Theta_0 + \gamma_0 \frac{M_2}{2!} + \gamma_1\left(x\frac{M_2}{2!} - \frac{M_3}{3!}\right) + \\ + \gamma_2\left(\frac{x^2}{2!} \cdot \frac{M_2}{2!} - x\frac{M_3}{3!} + \frac{M_4}{4!}\right) + \ldots \tag{20.16}$$

$$v = v_0 + \Theta_0 x + \gamma_0 \frac{M_3}{3!} + \gamma_1\left(x\frac{M_3}{3!} - 2\frac{M_4}{4!}\right) + \\ + \gamma_2\left(\frac{x^2}{2!} \cdot \frac{M_3}{3!} - 2x\frac{M_4}{4!} + 3\frac{M_5}{5!}\right) + \ldots \tag{20.17}$$

As $\gamma_3 = 0$ and $\gamma_4 = 0$ in the flexibility polynomial, formulas (20.16) and (20.17) may be limited to the terms containing γ_2.

The boundary conditions for the given beam are $x = 0$, $v_0 = 0$ and $x = 5l$, $v = 0$...

These conditions are employed to determine the initial values. First, the initial angle is determined from the condition at support B, according to (20.17)

$$\Theta_0 = \frac{C_1}{5lEI_C} \tag{20.18}$$

where C_1 is an expression containing higher moments when $x = 5l$.
Substitution of

$$v_0 = 0; \quad \Theta_0 = \frac{C_1}{5lEI_C} \quad \text{and} \quad x = \frac{5}{2}\, l$$

into (20.16) and (20.17) gives the unknown values Θ_0 and v.
The sign of $\Theta_{5l/2}$ (slope at mid-span) indicates whether the maximum deflection is located to the right-hand or to the left-hand of the mid-span; when $\Theta_{5l/2} = 0$ the maximum deflection is in the mid-span.

21. DETERMINATION OF DISPLACEMENTS BY DERIVATIVES OF FLEXIBILITY ANALYTICAL EXPRESSION

The previous section dealt with non-uniform beams and used an algebraic polynomial with a finite number of terms to express the flexibility.

Another method of designing non-uniform beams will be considered below. The flexibility may be expressed by any analytical function in terms of the abscissa, provided the only condition is satisfied, that is the function itself and all its derivatives are single-valued and finite within the interval examined.

The most simple formulas are derived in case the flexibility is expressed by an algebraic polynomial, as in the first method. Both methods will give the same formulas for this particular case, a slight difference appearing only in the arrangement of terms.

Rewriting formula (20.3) after the polynomial is expanded and the terms are arranged according to the order of moment we obtain

$$\Theta_u = \Theta_0 + \gamma_0 \frac{M_2}{2!} + \gamma_1 x\, \frac{M_2}{2!} + \gamma_2 \frac{x^2}{2!} \cdot \frac{M_2}{2!} +$$
$$+ \gamma_3 \frac{x^3}{3!} \cdot \frac{M_2}{2!} + \gamma_4 \frac{x^4}{4!}\; \frac{M_2}{2!} + \cdots$$

or

$$\Theta_u = \Theta_0 + \left(\gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2} + \gamma_3 \frac{x^3}{3!} + \gamma_4 \frac{x^4}{4!}\right) \times$$
$$\times \frac{M_2}{2} - \left(\gamma_1 + \gamma_2 x + \gamma_3 \frac{x^2}{2} + \gamma_4 \frac{x^3}{3!} + \cdots\right) \cdot \frac{M_3}{3!} + \cdots$$

Since

$$\left.\begin{aligned}\gamma_0+\gamma_1x+\gamma_2\frac{x^2}{2}+\gamma_3\frac{x^3}{3!}+\gamma_4\frac{x^4}{4!}+\dots&=\gamma\\ \gamma_1+\gamma_2x+\gamma_3\frac{x^2}{2!}+\gamma_4\frac{x^3}{3!}+\dots&=\gamma'\ \text{etc.,}\end{aligned}\right\}\qquad(21.1)$$

by introducing the values of the flexibility derivatives we arrive at the final formula for the rotation angle

$$\Theta_X=\Theta_0+\gamma\frac{M_2}{2!}-\gamma'\frac{M_3}{3!}+\gamma''\frac{M_4}{4!}-\gamma'''\frac{M_5}{5!}+\gamma^{\mathrm{IV}}\frac{M_6}{6!}-\dots\qquad(21.2)$$

Similarly, the beam deflection is expressed by

$$v_X=v_0+\Theta_0x+\gamma\frac{M_3}{3!}-2\gamma'\frac{M_4}{4!}+3\gamma''\frac{M_5}{5!}\qquad(21.3)$$

These formulas may be also directly derived by integrating the differential equation of the elasticity curve

$$\frac{d^2v}{dx}=\gamma M_1$$

by parts for an arbitrary flexibility expression.

For example, the angle is expressed by

$$\begin{aligned}\Theta_X&=\Theta_0+\int\gamma M_1\,dx=\Theta_0+\int\gamma d\left(\frac{M_2}{2!}\right)=\\ &=\Theta_0+\gamma\frac{M_2}{2!}-\int\gamma' d\left(\frac{M_3}{3!}\right)=\\ &=\Theta_0+\gamma\frac{M_2}{2!}-\gamma'\frac{M_3}{3!}+\int\gamma'' d\left(\frac{M_4}{4!}\right)=\\ &=\Theta_0+\gamma\frac{M_2}{2!}-\gamma'\frac{M_3}{3!}+\gamma''\frac{M_4}{4!}-\dots\end{aligned}\qquad(21.4)$$

Hence, formulas (21.2) and (21.4) coincide and may be used for the rigidity evaluation of beams which have a continuously varying section, the flexibility of which may be expressed by an arbitrary function in terms of x, as well as by an algebraic polynomial.

Repeated integration of (21.2) or (21.4) performed twice, three and four times, gives formula (21.3) and, subsequently, the formulas for the flexural curve area

$$\Omega_X=v_0x+\Theta_0\frac{x^2}{2!}+\gamma\frac{M_4}{4!}-3\gamma'\frac{M_5}{5!}+6\gamma''\frac{M_6}{6!}-10\gamma'''\frac{M_7}{7!}+\dots\qquad(21.5)$$

and the static moment of area Ω_X with respect to a vertical axis passing through point X is

$$\begin{aligned}S_X&=v_0\frac{x^2}{2!}+\Theta_0\frac{x^3}{3!}+\gamma\frac{M_5}{5!}-4\gamma'\frac{M_6}{6!}+\\ &+10\gamma''\frac{M_7}{7!}-20\gamma'''\frac{M_8}{8!}+\dots,\ \text{etc.}^{*}\end{aligned}\qquad(21.6)$$

---

* Numerical factors of the integral bimoments are taken from the first four columns of Table 18.1, according to the integration rules.

Let us apply the formulas derived in the present section to the following problem.

Consider a beam with a fixed left end A and a free right end B under the influence of a load shown in Fig. 21.1. We shall determine the rotation angle and deflection of the free end knowing that the transversal flexibility is expressed by a second order parabola

$$\gamma = \frac{1}{EI_0}\left[1-(1-n)\frac{(l-x)^2}{l^2}\right]$$

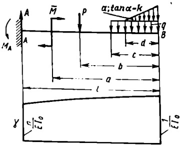

Fig. 21.1

First, the flexibility derivatives are determined by

$$\gamma' = \frac{2}{EI_0}(1-n)\frac{l-x}{l^2}$$

$$\gamma'' = -\frac{2}{EI_0}\cdot\frac{1-n}{l^2}; \quad \gamma''' = 0$$

When $x = l$ the flexibility and its derivatives at the free end are given by

$$\gamma = \frac{1}{EI_0}; \quad \gamma' = 0;$$

$$\gamma'' = -\frac{2}{EI_0}\cdot\frac{1-n}{l^2}$$

Finally, according to (21.2) and (21.3) we find

$$EI_0\Theta_B = \frac{M_2}{2} - (1-n)\frac{2}{l^2}\cdot\frac{M_4}{4!}$$

$$EI_0 v_B = \frac{M_3}{3!} - 3(1-n)\frac{2}{l^2}\cdot\frac{M_5}{5!}$$

Moments appearing in these formulas are determined from the general expression

$$\frac{M_n}{n!} = \left[-M_A\frac{l^{n-1}}{(n-1)!} + \bar{M}\frac{a^{n-1}}{(n-1)!}\right] + \left[\frac{Al^n}{n!} - \frac{Pb^n}{n!}\right] - q\frac{c^{n+1}}{(n+1)!} - k\frac{d^{n+2}}{(n+2)!}$$

Formulas are especially easy to apply when $x = 0$, i.e., for calculation of displacements at the origin of coordinates. This may be taken into account in evaluating simple beams and cantilevers.

Formulas (21.2) and (21.3)

$$\Theta = \Theta_0 + \gamma_u\frac{M_2}{2!} - \gamma'_u\frac{M_3}{3!} + \gamma''_u\frac{M_4}{4!} \tag{21.7}$$

$$v = x_0 + \Theta_0 x + \gamma_u\frac{M_3}{3!} - 2\gamma'_u\frac{M_4}{4!} + 3\gamma''_u\frac{M_5}{5!} \tag{21.8}$$

will be used to determine displacements in terms of flexibility derivatives for a beam by means of polynomial coefficients discussed in Sec. 20. The flexibility was previously given by

$$\gamma = \frac{1}{EI_C} i; \quad i = n + (1-n)\frac{4}{l}x - (1-n)\frac{8}{l^2}\cdot\frac{x^2}{2l}$$

hence,

$$EI_C\gamma'_u = (1-n)\frac{4}{l} - (1-n)\frac{8}{l^2}x$$

$$EI_C\gamma''_u = -(1-n)\frac{8}{l^2}$$

Formulas (21.7) and (21.8) are limited to terms containing γ''_u, since the derivatives of higher orders contain only γ_3, γ_4, which are zero in the present case [see formulas (21.1)].

Boundary conditions $v_0 = 0$ and $x = 5l$, $v = 0$ should be used to determine the initial angle Θ_0. This is performed by applying (21.8) to the right-hand end of the beam.

$$\Theta_0 = \frac{C_2}{5lEI_C} \tag{21.9}$$

where C_2 designates the expression containing higher moments when $x = 5l$.

After the initial angle Θ_0 is determined, and knowing that $v_0 = 0$, formulas (21.7) and (21.8) may be applied to determine angles and deflections of the beam. For example, at mid-span $\left(x = \frac{5}{2}l\right)$ formula (21.7) gives

$$\begin{aligned} EI_C\Theta_{\frac{5}{2}l} &= \frac{c_2}{5l} + \left\{n + (1-n)\frac{4}{l}\cdot\frac{5}{2}l - (1-n)\frac{8}{l}\left(\frac{5}{2}l\right)^2\frac{1}{2}\right\} \times \\ &\times \left\{A\left(\frac{5}{2}l\right)^2\frac{1}{2} - P\left(\frac{3}{2}l\right)^2\frac{1}{2} - q\left(\frac{l}{2}\right)^3\frac{1}{6}\right\} - \\ &- \left\{(1-n)\frac{4}{l} - (1-n)\frac{8}{l^2}\cdot\frac{5}{2}l\right\}\left\{A\left(\frac{5}{2}l\right)^3\frac{1}{6} - \ldots\right. \end{aligned} \tag{21.10}$$

Similarly, according to (21.8)

$$\begin{aligned} EI_Cv_{\frac{5}{2}l} &= \frac{c_2}{5l}\cdot\frac{5l}{2} + \left\{n + (1-n)\frac{4}{l}\cdot\frac{5l}{2} - (1-n)\frac{8}{l^2}\frac{(5l/2)^2}{2}\cdot\frac{1}{2}\right\} \times \\ &\times \left\{A\frac{(5l/2)^3}{6} - P\frac{(3l/2)^3}{6} - q\frac{(l/2)^4}{24}\right\} - \\ &- 2\left\{(1-n)\frac{4}{l} - (1-n)\frac{8}{l^2}\cdot\frac{5l}{2}\right\} \times \\ &\times \left\{A\frac{(5l/2)^4}{24} - P\frac{(3l/2)^4}{24} - q\frac{(l/2)^5}{120}\right\} - \\ &- 3\left\{(1-n)\frac{8}{l^2}\right\}\left\{A\frac{(5l/2)^5}{120} - P\frac{(3l/2)^5}{120} - q\frac{(l/2)^6}{720}\right\} \end{aligned} \tag{21.11}$$

22. DETERMINATION OF DISPLACEMENTS BY FLEXIBILITY INTEGRALS

In Sec. 21 the method of computation was described, in which formulas (21.2)-(21.4) were obtained by integrating the flexibility equation presented as a function of abscissa x. In the resulting series the first integral expresses the rotation angle while the subsequent integrals correspond to other quantities.

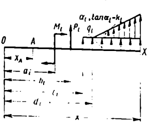

Fig. 22.1

The order of the moments in these polynomials increases with every term according to the usual integration rules and every term contains as factors successive derivatives of the beam flexibility.

Subsequent integration may be performed in another way which will lead to a series of terms representing the derived products of the integral moments of the beam transverse flexibility.

Let Γ with corresponding subscripts denote successive integrals of the flexibility γ_0, so that*

$$\Gamma_1 = \int \gamma\, dx; \quad \Gamma_2 = \int\int \gamma\, dx^2; \quad \Gamma_n = \overbrace{\int \cdots \int}^{n \text{ times}} \gamma\, dx^n \qquad (22.1)$$

When integrating the differential equation of the flexural curve we take into account that $\frac{d^2v}{dx^2} = M\gamma$

$$\frac{d\Gamma_1}{dx} = \gamma; \quad \frac{d\Gamma_2}{dx} = \Gamma_1; \quad \frac{d\Gamma_3}{dx} = \Gamma_2, \text{ etc.}$$

As the order of the moments decreases, the expression obtains terms with moments of order lower than unity. Such moments are discontinuous functions. Discontinuity appears at points where a certain load is applied corresponding to the order of the discontinuous moment. For example, discontinuity appears for a moment of order $n = 1$ at the point where force couples are applied; for a moment of order $n = 0$ it is at the point of application of point forces; for a moment of order $n = -1$ it is at the beginning of a uniformly distributed load (when passing along the beam from the left to the right); for a moment of order $n = -2$ it is at the point where a uniformly changing load begins, etc. (Fig. 22.1).

All these lower order moments are discontinuous at the above mentioned positions but are continuous to the right of these points.

---

* Multiple integrals may be replaced by moments or by ordinary integrals.

Integration and differentiation of such functions may be performed if integrals are considered to be of the Stieltjes type, not the usual integrals of continuous functions (Riman's integrals).

Thus, we arrive at the following formulas for the rotation angle

$$\Theta_X = \Theta_A + \int_0^x M_1 \gamma \, du = \Theta_A + \int_0^x M_1 \, d\Gamma_1 = \Theta_A + [M_1\Gamma_1]_{u=0}^{u=x} -$$

$$- \int_0^x \Gamma_1 \frac{dM_1}{dx} \, dx = \Theta_A + [M_1\Gamma_1]_{u=0}^{u=x} - \int_0^x \Gamma_1 M_0 \, dx =$$

$$= \Theta_A + [M_1\Gamma_1]_{u=0}^{u=x} - \int_0^x M_0 \, d\Gamma_2 = \ldots$$

Since

$$M_0 = \frac{dM_1}{dx} = Q_X; \quad M_{-1} = \frac{dM_0}{dx} = \frac{dQ_X}{dx} = q_X \text{ and}$$

$$M_{-2} = \frac{dq_X}{dx} = k_X$$

we find

$$\Theta_X = \Theta_A + [M_1\Gamma_1]_{n=0}^{n=x} - \int_0^x Q \, d\Gamma_2 = \ldots$$

As successive integration is carried out, the bending moment, the transverse force, the load intensity and the angular coefficient of the loading area (i.e., $M_1 = M_X$; $M_0 = Q_X$; $M_{-1} = q_X$; $M_{-2} = k_X$) form successive quantities, every member being a derivative of the preceding one

$$\Theta_X = \Theta_A + [M_1\Gamma_1]_{u=0}^{u=x} - [M_0\Gamma_2]_{u=0}^{u=x} + [M_{-1}\Gamma_3]_{u=0}^{u=x} - [M_{-2}\Gamma_4]_{u=0}^{u=x} \quad (22.2)$$

Since no parabolic loads will be examined in the further discussion, i.e., only linear types of loading (such as rectangular, triangular and trapezoidal) are considered, the series in (22.2) is limited to the term $[M_{-2}\Gamma_{-4}]$, because

$$M_{-3} = \frac{dk_X}{dx} = 0$$

Substitution of upper and lower limits of variable u in (22.2) must be performed with the discontinuity of $M_1 = M_X$; $M_0 = Q_X$; $M_{-1} = q_X$; $M_{-2} = k_X$ taken into account, as mentioned previously. This gives a step in the bending moment diagram at the point where the force couple is applied. If several force couples are applied to the beam, then the bending moment diagram will contain as many steps along the abscissa x as the number of force couples. Similarly, steps will appear in the transverse force diagram equal to the corresponding point loads at sections where these loads are applied.

The distributed load diagram will contain steps at the beginning of each distributed load. Finally, the diagram of the intensity derivatives $\left(M_{-2} = k_X = \frac{dq_X}{dx} = \frac{d^2Q_X}{d_x}\right)$ will contain steps at the points where the loading line breaks.

For generalization, a beam will be considered loaded by a combination of force couples, point loads, uniformly distributed loads and uniformly increasing loads according to Fig. 22.1, which shows only one load of each type for simplicity.

Substitution of the integration limits from (22.2) gives

$$\Theta_X = \Theta_A + [M_X\Gamma_{1.\,X} - Q_X\Gamma_{2.\,X} + q_X\Gamma_{3.\,X} - k_X\Gamma_{4.\,X}] -$$

$$- [\sum \overline{M}_l\Gamma_{1,\,a_l} - \sum P_l\Gamma_{2,\,b_l} + \sum Q_l\Gamma_{3,\,c_l} - \sum k_l\Gamma_{4,\,d_l}] \quad (22.3)$$

where M_X, Q_X, q_X, k_X = bending moment, transverse force, loading line ordinate and its derivative, respectively, at section X with abscissa x, in which the rotation angle is determined

$\Gamma_{1.\,X}$, $\Gamma_{2.\,X}$, $\Gamma_{3.\,X}$, $\Gamma_{4.\,X}$ = first four integrals of the flexibility at the same section*

$\Gamma_{1,\,a_l}$ = first integral of the transverse flexibility at $x = a_l$ where the force couple M_l is applied

$\Gamma_{2,\,b_l}$ = second integral of the transverse flexibility at $x = b_l$ where the point force P_l is applied, etc.

The values of M_X, Q_X, q_X and k_X are determined by formulas of Sec. 7 or as particular cases of formula (I), Sec. 6.

Integrals are especially easy to calculate if the flexibility is expressed by an algebraic polynomial

$$\gamma = \gamma_0 + \gamma_1 u + \gamma \frac{u^2}{2} + \gamma_3 \frac{u^3}{3} + \ldots = \sum_{i=0,\,1,\,2\ldots} \gamma_i \frac{u^i}{i!}$$

$$\Gamma_{1.\,u} = \sum_{i=0,\,1,\,2\ldots} \gamma_i \frac{u^{i+1}}{(i+1)!}; \quad \Gamma_{2.\,u} = \sum_{i=0,\,1,\,2\ldots} \gamma_i \frac{u^{i+2}}{(i+2)!}$$

or in the general form

$$\Gamma_{n.\,u} = \sum_{i=0,\,1,\,2\ldots} \gamma_i \frac{u^{i+n}}{(i+n)!} \quad (22.4)$$

---

* The first subscript (the figure) of the Γ symbol corresponds to the integration order, while the second subscript (the letter) gives the abscissa of the section at which the corresponding integral of the flexibility is determined.

If point X along abscissa x coincides with point U with abscissa u, the rotation angle is found from formula (22.3)[*]

$$\Theta_U = \Theta_A + [M_U\Gamma_{1,u} - Q_U\Gamma_{2,u} + q_U\Gamma_{3,u} - k_U\Gamma_{4,u}] - \\ - [\sum \overline{M}_i\Gamma_{1,a_i} - \sum P_i\Gamma_{2,b_i} + \sum q_i\Gamma_{3,c_i} - \sum k_i\Gamma_{4,d_i}] \quad (22.5)$$

Integrating formula (22.5) we must take into consideration th way of deriving (22.3)

$$v_X = v_A + \int\limits_{u=0}^{u=x} \Theta_U\,du = v_A + \Theta_A x + [M\Gamma_2 - 2Q\Gamma_3 + \\ + 3q\Gamma_4 - 4k\Gamma_5]_{u=0}^{u=x} - \Big\{\Big[\sum \overline{M}_i\Gamma_{1,a_i} - \sum P_i\Gamma_{2,b_i} + \\ + \sum q_i\Gamma_{3,c_i} - \sum k_i\Gamma_{4,d_i}\Big]u\Big\}_{u=0}^{u=x}$$

which transforms after the integration limits are substituted into

$$v_X = v_A + \Theta_A x + [M_X\Gamma_{2,x} - 2Q_X\Gamma_{3,x} + 3q\Gamma_{4,x} - 4k\Gamma_{5,x}] - \\ - [\sum \overline{M}_i\Gamma_{2,a_i} - 2\sum P_i\Gamma_{3,b_i} + 3\sum q_i\Gamma_{4,c_i} - 4\sum k_i\Gamma_{5,d_i}] - \\ - [\sum \overline{M}_i\Gamma_{1,a_i}(x-a_i) - \sum P_i\Gamma_2, b_i(x-b_i) + \sum q_i\Gamma_{3,c_i}(x-c_i) - \\ - \sum k_i\Gamma_{4,d_i}(k-d_i)] \quad (22.6)$$

The structure of these formulas becomes clear after examining a compound load (Fig. 22.1). Moments of negative order cannot be obtained as particular cases from the general formula of the n-th moment because for point loads it would mean raising the moment arm to a negative power.

To elucidate further presentation, here is a summary of moments with only non-vanishing terms included; the following formulas relate to $n = 3, 2, 1, 0, -1$ and -2, respectively,

$$\left.\begin{aligned} &\frac{\overline{M}_3}{3!} = \overline{M}\,\frac{(x-a)^2}{2} + P\,\frac{(x-b)^3}{3!} + q\,\frac{(x-c)^4}{4!} + k\,\frac{(x-d)^5}{5!} \\ &\frac{M_2}{2} = \overline{M}\,(x-a) + P\,\frac{(x-b)^2}{2} + q\,\frac{(x-c)^3}{3!} + k\,\frac{(x-d)^4}{4!} \\ &\frac{M_1}{1!} = \overline{M} + P\,(x-b) + q\,\frac{(x-c)^2}{2} + k\,\frac{(x-d)^3}{3!} \\ &M_0 = P + q\,(x-c) + k\,\frac{(x-d)^2}{2} \\ &M_{-1} = q + k\,(x-d); \;\; M_{-2} = k \end{aligned}\right\} \quad (22.7)$$

---

[*] In order to avoid any possible mistakes we must point out that q_U and k_U are the second and third derivatives of the bending moment at a section U with abscissa u (i.e., the ordinate of the loading line and the angular coefficient of its tangent, or the first and second derivatives of the transverse force), while q_i is the intensity of the i-th distributed load beginning from the section with abscissa c_i; k_i is the angular coefficient of the i-th triangular loaded area the apex of which points to the left and beginning from the section with abscissa d_i. The abscissa axis points to the right, as usual, and its origin is at the left end of the beam.

Thus, moment M_1 is of the lowest possible order to include all loads shown in Fig. 22.1. The zero moment M_0 includes only forces but not force couples. The moment M_{-1} corresponds to the intensity of distributed loads and its expression contains neither $\overline{M}$ nor P. Finally, in the expression for $\overline{M}_{-2}$ which represents the rate of continuous load variation, only k is included.

Thus, all moments of orders lower than -2 have no physical meaning and since M_{-3} and M_{-4} vanish for loads shown in Fig. 22.1, the series obtained in formulas (22.3) and (22.6) are limited to the terms containing M_{-2}.

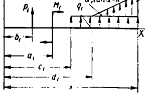

Fig. 22.2

These series are even shorter if the loads are simpler than those shown in Fig. 22.1.

For example, if only point loads are applied to a beam, then the polynomials for Θ and v are limited to terms containing M_0. *Thus, when moments of negative order are used in deriving formulas for Θ and v, the series of moments are finite and solutions obtained are exact for any compound load, while the two previously described methods give approximate solutions* *.

The simpler the load applied to the beam, the simpler is the expression for the exact solution. This simplification concerns not the number of terms corresponding to the same type of loading, but to the type of loading itself. For example, formulas become simpler if there are no non-uniform distributed loads or no distributed loads at all.

Additional explanations are required for the application of integrals Γ_1, Γ_2, Γ_3. These integrals were given by (22.4) for the case where the flexibility γ can be expressed by an algebraic polynomial.

In case the rigidity of a beam is expressed by a monomial $EI_u = gu^n$, the flexibility is given by

$$\gamma_u = \frac{1}{EI_u} = \frac{1}{g} u^{-n}$$

which leads to the following expressions for the integrals:

$$\left.\begin{aligned} \Gamma_1 &= -\frac{1}{g} \cdot \frac{u^{-n+1}}{n-1} \\ \Gamma_2 &= \frac{1}{g} \cdot \frac{u^{-n+2}}{(n-1)(n-2)} = \frac{1}{g} u^{-n+2} \frac{(n-3)!}{(n-1)!} \\ \Gamma_3 &= -\frac{1}{g} u^{-n+3} \frac{(n-4)!}{(n-1)!}; \quad \Gamma_4 = \frac{1}{g} u^{-n+4} \frac{(n-5)!}{(n-1)!} \end{aligned}\right\} \qquad (22.8)$$

---

* Solutions are exact if the flexibility of a beam along its axis is presented by an exact analytical expression, of which integrals Γ_1, Γ_2, Γ_3, . . . are taken.

Problem 1. Let a simple beam of a non-uniform section be in flexure under the influence of the load given in Fig. 22.2. Determine the angles Θ_A and Θ_B at the supports and the angle and deflection at section X with abscissa x.

The integration constant $\Theta_0 = \Theta_A$ is determined from (22.6) by assuming $x = l$ and by substituting into (22.6) the following static factors

$$\left.\begin{aligned} &M_B = 0; \quad Q_B = -B \\ &q_B = -q - k(l-d); \quad k_B = -k \end{aligned}\right\} \tag{22.9}$$

which gives the equation

$$\begin{aligned} \Theta_A l &+ \{2B\Gamma_{3,\,l} - 3[q + k(l-d)]\,\Gamma_{4,\,l} + 4k\Gamma_{5,\,l}\} - \\ &- [\overline{M}\Gamma_{2,\,a} + 2P\Gamma_{3,\,b} - 3q\Gamma_{4,\,c} + 4k\Gamma_{5,\,d}] - \\ &- [\overline{M}\Gamma_{1,\,a}(l-a) + P\Gamma_{2,\,b}(l-b) - q\Gamma_{3,\,c}(l-c) + \\ &+ k\Gamma_{4,\,d}(l-d) = 0 \end{aligned} \tag{22.10}$$

from which we determine Θ_A*.

Formula (22.3) gives the following expression for the angle

$$\begin{aligned} \Theta_X = \Theta_A &+ \left[Ax - \overline{M} - P(x-b) - q\frac{(x-c)^2}{2} - k\frac{(x-d)^3}{3!}\right]\Gamma_{1,\,x} - \\ &- \left[A - P - q(x-c) - k\frac{(x-d)^2}{2}\right]\Gamma_{2,\,x} + \\ &+ [-q - k(x-d)]\,\Gamma_{3,\,x} + k\Gamma_{4,\,x} - \\ &- \{\overline{M}\Gamma_{1,\,a} + P\Gamma_{2,\,b} - q\Gamma_{3,\,c} + k\Gamma_{4,\,d}\} \end{aligned} \tag{22.11}$$

Assuming $x = l$ and noting that $M_B = 0$ we arrive at the following formula for the angle at support B

$$\begin{aligned} \Theta_B = \Theta_A &+ \{B\Gamma_{2,\,l} + [-q + k(l-d)]\,\Gamma_{3,\,l} + k\Gamma_{4,\,l}\} - \\ &- \{\overline{M}\Gamma_{1,\,a} + P\Gamma_{2,\,b} - q\Gamma_{3,\,c} + k\Gamma_{4,\,d}\} \end{aligned} \tag{22.12}$$

Formula (22.6) is applied to determine the deflection at section X

$$\begin{aligned} v_X = \Theta_A x &+ \Big\{\left[Ax - \overline{M} - P(x-b) - q\frac{(x-c)^2}{2} - \right. \\ &\left. - k\frac{(x-d)^3}{3!}\right]\Gamma_{2,\,x} - 2\left[A - P - q(x-c) - k\frac{(x-d)^2}{2}\right]\Gamma_{3,\,x} + \\ &+ 3[-q - k(x-d)]\,\Gamma_{4,\,x} + 4k\Gamma_{5,\,x}\Big\} - [\overline{M}\Gamma_{2,\,a} + \\ &+ 2P\Gamma_{3,\,b} - 3q\Gamma_{4,\,c} + 4k\Gamma_{5,\,d}] - [\overline{M}\Gamma_{1,\,a}(x-a) + \\ &+ P\Gamma_{2,\,b}(x-b) - q\Gamma_{3,\,c}(x-c) + k\Gamma_{4,\,d}(x-d)] \end{aligned} \tag{22.13}$$

---

* In deriving (22.10) from (22.6) the value of the given loading was considered. The force couple was assumed positive, while the other loads are negative according to the sign rules established previously. As all the integrals Γ are zero when $u = 0$, the reaction A is not included in (22.10).

Integrals Γ_1, Γ_2, Γ_3, Γ_4 for substitution into formulas (22.11)-(22.12) are determined from (22.4) when the flexibility is expressed by an algebraic polynomial.

The problem solved in Sec. 20 in terms of polynomial coefficients will be solved now in terms of integrals of the same flexibility polynomial.

When

$$EI_c\gamma = n + (1-n)\frac{4}{l}x - (1-n)\frac{8}{l^2}\cdot\frac{x^2}{2}$$

the integrals are expressed by the following formulas

$$\left.\begin{aligned} EI_c\Gamma_1 &= nx + (1-n)\frac{4}{l}\cdot\frac{x^2}{2} - (1-n)\frac{8}{l^2}\cdot\frac{x^3}{6} \\ EI_c\Gamma_2 &= n\frac{x^2}{2} + (1-n)\frac{4}{l}\cdot\frac{x^3}{6} - (1-n)\frac{8}{l^2}\cdot\frac{x^4}{24} \\ EI_c\Gamma_3 &= n\frac{x^3}{6} + (1-n)\frac{4}{l}\cdot\frac{x^4}{24} - (1-n)\frac{8}{l^2}\cdot\frac{x^5}{120} \end{aligned}\right\} \qquad (22.14)$$

Using these expressions to solve the problem, according to (22.2), we obtain

$$\Theta_x = \Theta_0 + [\Gamma_1 M_1 - \Gamma_2 M_0 + \Gamma_3 M_{-1} - \Gamma_4 M_{-2}]_{u=0}^{u=x} \qquad (22.15)$$

Since all moments of orders below -2 are zero for the given load, formula (22.15) contains only moments of order M_{-2} and higher.

Formula (22.15) gives the exact solution; its comparison with the solutions obtained by the previous formulas (in terms of flexibility polynomial coefficients and in terms of derivatives of the flexibility) makes it possible to evaluate the accuracy of previously performed calculations if the latter were approximate, i.e., if the flexibility was expressed by a polynomial or an infinite series limited to the second, third, ... members.

First, the initial angle is determined as previously from the boundary conditions that $v_0 = 0$ when $x = 0$ and the deflection $v = 0$ when $x = 5l$.

Assuming $x = 5l$ in (22.6) we arrive at the following expression for the deflection

$$\begin{aligned} EI_C v_x = {} & EI_C v_0 + EI_C\Theta_0 x + [\Gamma_2 M_1 - 2\Gamma_3 M_0 + 3\Gamma_4 M_{-1} - 4\Gamma_5 M_{-2}] - \\ & - [\Gamma_2\overline{M} - 2\Gamma_3(-P) - 2\Gamma_3 A + 3\Gamma_4(-q) - 4\Gamma_5 k] - \\ & - [\Gamma_1\overline{M} - \Gamma_2(-P) - \Gamma_2(A) + \Gamma_3(-q) - \Gamma_4 k]\,x + \\ & + \Gamma_1\overline{M}x_M - \Gamma_2(-P)\,x_p + \Gamma_3(-q)\,x_q - \Gamma_4 k x_k \end{aligned} \qquad (22.16)$$

where $x_M = 3l$, $x_p = l$, $x_q = 2l$, $x_k = 4l$ according to the given load and all terms containing A vanish because $\Gamma_2 = 0$ and $\Gamma_3 = 0$ when $x = 0$.

From formula (22.16) we find Θ_0 when $x = 5l$. Next, formulas (22.15) and (22.16) determine the angle and the deflection when $x = \frac{5}{2} l$.

Formulas given in the present section are polynomial because the beam is assumed to be subjected to a rather complicated compound load. Formulas presented in Chap. IV make it possible to obtain solutions without performing any integration. It is only necessary to substitute certain quantities into the formulas according to the given load; further algebraic operations are similar to the ones performed for prismatic beams.

All the formulas in this section can be presented in other forms based on (7.11). Integrals of the flexibility may be replaced by higher moments of the flexibility diagram, if the following relationship is taken into account

$$\Gamma_n = \frac{M_{n-1}}{(n-1)!} \tag{22.17}$$

Substituting $n=1, 2, 3, 4, 5$, we obtain

$$\Theta_X = \Theta_A + \left[M_X M_{X^0} - Q_X M_{X'} + q_X \frac{M_{X^2}}{2!} - k_X \frac{M_{X^3}}{3!} \right] - \\ - \left[\sum M_i M_{a_i^0} - \sum P_i M_{b_2'} + \sum q_i \frac{M_{c_i^2}}{2!} - \sum k_i \frac{M_{d_i^3}}{3!} \right] \tag{22.18}$$

and

$$v_X = v_A + \Theta_A x + \left[M_X M_{X'} - 2Q_X \frac{M_{X^2}}{2!} + 3q_X \frac{M_{X^3}}{3!} - 4k_X \frac{M_{X^4}}{4!} \right] - \\ - \left[\sum M_i M_{a_i'} - 2 \sum P_i \cdot \frac{M_{b_i^2}}{2!} + 3 \sum q_i \frac{M_{c_i^3}}{3!} - 4 \sum k_i \frac{M_{d_i^4}}{4!} \right] - \\ - \left[\sum \overline{M}_i M_{a_i^0} (x - a_i) - \sum P_i M_{b_i'} (x - b_i) + \right. \\ \left. + \sum q_i \frac{M_{c_i^2}}{2} (x - c_i) - \sum k_i \frac{M_{d_i^3}}{3!} (x - d_i) \right] \tag{22.19}$$

where M_X, Q_X, q_X, k_X = bending moment, transverse force, distributed load intensity, and derivative of intensity with respect to the abscissa $\left(k_X = \frac{dq_X}{dx}\right)$ determined at the section with abscissa x

M_{X^0}, M_{X^1}, M_{X^2}, M_{X^3}, M_{X^4} = moments of zero, first, second, third and fourth orders of the diagram area to the left of point X with respect to this point

$\overline{M}_i$, P_i, q_i, k_i = force couple, point load, intensity of the uniformly distributed load, and the angular coefficient of the triangular loaded area with its apex pointing to the left (Fig. 22.3)

$M_{a_i^0}$, $M_{a_i^1}$ = moments of zero and first orders of the flexibility diagram to the left of point X with respect to the point of application of the force couple with abscissa a_i

$M_{b_i^1}$, $M_{b_i^2}$ = moments of first and second order of the flexibility diagram to the left of point X with respect to the point of application of the point load with abscissa b_i

$M_{c_i^2}$, $M_{c_i^3}$ = moments of second and third order of the flexibility diagram to the left of point X with respect to the left point of application of the uniformly distributed load with abscissa c_i

$M_{d_i^3}$, $M_{d_i^4}$ = moments of third and fourth order of the flexibility diagram to the left of point X with respect to the point where the apex of triangular load is located, from which this load increases until it reaches the right-hand end of the beam, with abscissa d_i.

Problem 2. Examine a non-uniform beam under the influence of a compound load which consists of the following components (Fig. 22.4): (1) point loads P_1, P_2, P_3 applied at sections with abscissas b_1, b_2, b_3, respectively (the first and the third forces are considered negative, while the second is positive); (2) a negative uniformly distributed load of intensity $(-q)$ which begins at the section with abscissa c; (3) a positive uniformly increasing load with an angle coefficient $k = \tan\alpha$ which begins at the section with abscissa a.

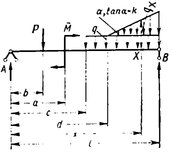

Fig. 22.3

Let the flexibility be given by

$$\gamma = \frac{1}{EI} = \frac{1}{EI_C} \cdot \frac{EI_C}{EI} = \frac{1}{EI_C}\, i \qquad (22.20)$$

where EI_C is the rigidity of the beam at mid-span.

The variable i is shown in Fig. 22.4; its diagram is outlined by a hexagonal with ordinates equal to unity in the central part of the beam and ordinates n at the end supports. The left-hand straight line $A'E'$ is expressed by

$$i = 1 - (1-n)\frac{u-x}{u} \qquad (22.21)$$

hence, $i = n$ when $x = 0$ and $i = 1$ when $x = u$.

The rotation angles at the supports are determined by formulas (22.19) and (22.18); the first two methods given in Sec. 20 and 21 cannot be applied to this problem because the transverse flexibility varying along the span follows three different laws.

According to formula (22.19)

$$\begin{aligned} EI_C v_B = {} & EI_C \Theta_A l + \Big[-2Q_B \frac{M_{B^2}}{2} + \\ & + 3q_B \frac{M_{B^3}}{3!} - 4k_B \frac{M_{B^4}}{4!}\Big] - \\ & - \Big[-2(-P_1)\frac{M_{b_1^2}}{2} - \\ & - 2(+P_2)\frac{M_{b_2^2}}{2} - 2(-P_3)\frac{M_{b_3^2}}{2} + \\ & + 3(-q)\frac{M_{c^3}}{3!} - 4(+k)\frac{M_{d^4}}{4!}\Big] - \\ & - \Big[-(-P_1)(x-b_1)M_{b_1^1} - (+P_2) \times \\ & \times (x-b_2)M_{b_2^1} - (-P_3)(x-b_3) \times \\ & \times M_{b_3^1} + (-q)(x-c)\frac{M_{c^2}}{2} - (+k) \times \\ & \times (x-d)\frac{M_{d^3}}{3!}\Big] = 0 \end{aligned} \qquad (22.22)$$

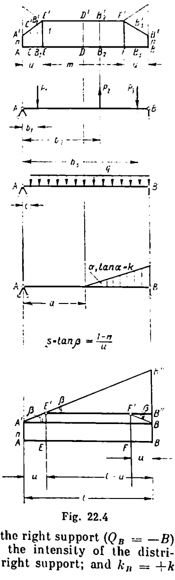

Fig. 22.4

where Q_B = transverse force at the right support ($Q_B = -B$)
$q_B = -q + k(l-d)$ is the intensity of the distributed load at the right support; and $k_B = +k$

M_{B^2}, M_{B^3}, M_{B^4} = moments of second, third and fourth orders of the total diagram area i with respect to the vertical axis passing through the right support

$M_{b_1^2}$ = second moment of the left-hand part of diagram i with respect to point B, i.e., the area $AA'B'B_1$

$M_{b_2^2}$, $M_{b_3^2}$ = second moments of the left-hand part of diagram i with respect to points B_2 and B_3, etc.

We must remember that the letter of the subscript indicates the moment axis while the exponent figure indicates the order of the moment.

To calculate higher moments of the diagram with respect to the right-hand end of the beam, the diagram is considered to consist of four figures: (1) rectangle $AA'B'B$ of length l and height n; (2) triangle $A'B'''B'$ of length l and slope $s = \tan\beta = \frac{1-n}{u}$; (3) negative triangle $E'B'''B''$ of length $(l-u)$ and the same slope; (4) negative triangle $F'B''B'$ of length u and the same (though negative) slope.

The above considerations lead to the following general formula

$$\frac{M_{B^m}}{m!} = n\frac{l^{m+1}}{(m+1)!} + \frac{1-n}{u}\cdot\frac{l^{m+2}}{(m+2)!} - \frac{1-n}{u}\cdot\frac{(l-u)^{m+2}}{(m+2)!} - \frac{1-n}{u}\cdot\frac{u^{m+2}}{(m+2)!} \tag{22.23}$$

This method may be applied not only to the right-hand part but also to the left-hand part of any section; then we calculate higher moments for substitution into formula (22.22).

Finding Θ_A from (22.22) and substituting it into (22.18) we obtain the expression for angular displacement Θ_B on the right-hand diagram. Then

$$\begin{aligned} EI_C\Theta_B = EI_C\Theta_A + \left[-Q_B M_{B^1} + q_{B^1}\frac{M_{B^2}}{2!} - k_B\frac{M_{B^3}}{3!}\right] - \\ -\Big[-(-P_1)M_{b_1^1} - (+P_2)M_{b_2^1} - (-P_3)M_{b_3^1} + \\ +(-q)\frac{M_{c^2}}{2} - (+k)\frac{M_{d^3}}{3!}\Big] \end{aligned} \tag{22.24}$$

Problem 3. Consider a beam with a fixed left-hand end under the influence of a compound load (Fig. 22.5).

The flexibility is expressed by

$$\frac{1}{EI} = \frac{1}{EI_C}\cdot\frac{EI_C}{EI} = \frac{1}{EI_C}i$$

where EI_C is the lowest rigidity at the free end of the beam.

Let the flexibility be expressed by a parabolic function, so that

$$i = 1-(1-n)\frac{(l-x)^2}{l^2} = n + \frac{2(l-n)}{l}x - \frac{2(l-n)}{l^2}\cdot\frac{x^2}{2!} \tag{22.25}$$

Then in the trinomial expression for the flexibility

$$\gamma=\gamma_0+\gamma_1 x+\gamma_2\frac{x^2}{2!} \tag{22.26}$$

the coefficients are given by

$$\gamma_0=\frac{n}{EI_C}; \quad \gamma_1=\frac{2(1-n)}{lEI_C}; \quad \gamma_2=-\frac{2(l-n)}{l^2EI_C} \tag{22.27}$$

From (22.26) $\gamma'=\gamma_1+\gamma_2 x$; $\gamma''=\gamma_2$

Therefore

$$\gamma'=\frac{2(1-n)}{lEI_C}-\frac{2(1-n)}{l^2EI_C}x; \quad \gamma''=-\frac{2(1-n)}{l^2EI_C} \tag{22.28}$$

According to (22.26), the i diagram may be considered to consist of three parts, a rectangle with an ordinate $n=EI_C\gamma_0$, a triangle with a slope $\tan\beta=EI_C\gamma_1$ and a parabola. This means that *higher moments of the flexibility diagram to the left of point X or integrals of the flexibility* (which is essentially the same) may be given by the general formula

$$\Gamma_m=\frac{M_{X^{m-1}}}{(m-1)!}=\gamma_0\frac{x^m}{m!}+\gamma_1\frac{x^{m+1}}{(m+1)!}+$$
$$+\gamma_2\frac{x^{m+2}}{(m+2)!}+\dots \tag{22.29}$$

Substituting $m=1, 2, 3, 4$ into (22.29) we obtain $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, respectively.

The angular displacement at point B is determined by

$$\Theta_B=\gamma_0\frac{M_{B^2}}{2}+\gamma_1\left(l\frac{M_{B^2}}{2}-\frac{M_{B^3}}{3!}\right)+$$
$$+\gamma_2\left(\frac{l^2}{2}\cdot\frac{M_{B^2}}{2}-l\frac{M_{B^3}}{3!}+\frac{M_{B^4}}{4!}\right) \tag{22.30}$$

in which γ_0, γ_1 and γ_2 are given by (22.27) and from (22.30) the higher moments of the load are expressed by

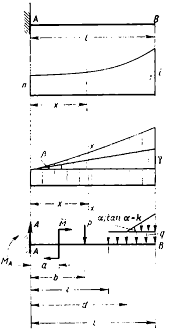

Fig. 22.5

$$\frac{M_{B^m}}{m!}=\left[M_A\frac{l^{m-1}}{(m-1)!}+\overline{M}\frac{(l-a)^{m-1}}{(m-1)!}\right]+$$
$$+\left[A\frac{l^m}{m!}-P\frac{(l-b)^m}{m!}\right]-\frac{q(l-c)^{m+1}}{(m+1)!}-k\frac{(l-d)^{m+2}}{(m+2)!} \tag{22.31}$$

by substituting $m=1, 2, 3$, respectively.

The same angle may be determined by

$$\Theta_B = \gamma_B \frac{M_{B^2}}{2} - \gamma'_B \frac{M_{B^3}}{3!} + \gamma''_B \frac{M_{B^4}}{4!} \tag{22.32}$$

where the moments are given by (22.31) and the flexibility and its derivatives are calculated from (22.26) and (22.28) according to which

$$\gamma_B = \frac{1}{EI_C}; \quad \gamma'_B = 0; \quad \gamma''_B = -\frac{2(1-n)}{l^2 EI_C} \tag{22.33}$$

This particular problem is preferably solved in terms of derivatives of the flexibility than by the second calculation method.

In determining the angle through the integrals of flexibility, formula (22.18) may be used

$$\Theta_B = \left[-k_B \frac{M_{B^3}}{3!} \right] - \left[(-\overline{M}) M_{a^0} - (-P) M_{b^1} + \right.$$
$$\left. + (-q) \frac{M_{c^2}}{2} - (-k) \frac{M_{d^3}}{2!} \right] \tag{22.34}$$

where the signs of the loads are negative and the boundary conditions $M_B = 0$ and $Q_B = 0$ are taken into account; in addition, from Fig. 22.5 we see that

$$q_B = -q + k(l - d); \quad k_B = -k$$

Higher moments of the flexibility diagram to be substituted into (22.34) are given by (22.29). To calculate these, the value of x must be equal to the corresponding abscissa; thus, $x = l$ for M_{B^3}; $x = d$ for M_{a^0}; $x = b$ for M_{b^1}; $x = c$ for M_{c^2}; and $x = d$ for M_{d^3}.

These calculations appear to be more cumbersome than in the second method, described in Sec. 21. That is why the deflection of the free end is determined only by the second method. In general, the second method is preferable for non-uniform beams when the flexibility is expressed by an analytical law (Sec. 21); however, if the flexibility diagram is outlined by a polygon (Fig. 22.4) or by several curves, the third method may be more convenient (Sec. 22).

23. DETERMINATION OF DISPLACEMENTS WHEN RIGIDITY FOLLOWS POWER LAW

Subsidiary formulas for integral bimoments derived in Sec. 19 cannot be applied directly if the abscissa x in the initial bimoment expression is raised to a negative power.

Supplementary formulas will be derived herein for calculating non-uniform beams, the rigidity of which is expressed by a power law.

Let the initial bimoment be given by

$$y=(m-1)!\,x^{-m}\frac{M_n}{n!} \tag{23.1}$$

where the factorial $(m-1)!$ is included to obtain a definite regularity in the sequence of terms of the integral formulas. Then the first integral is

$$I_1=\int y\,dx=(m-1)!\int x^{-m}\frac{M_n}{n!}\,dx=(m-1)!\int\frac{M_n}{n!}\,d\frac{x^{-m+1}}{-m+1}=$$
$$=-(m-2)!\,x^{-m+1}\frac{M_n}{n!}+(m-2)!\int x^{-m+1}\frac{M_{n-1}}{(n-1)!}\,dx=\dots$$

After multiple integration by parts we obtain

$$\begin{aligned}I_1&=(m-1)!\int x^{-m}\frac{M_n}{n!}\,dx=\\&=-(m-2)!\frac{M_n}{n!}x^{-m+1}-(m-3)!\frac{M_{n-1}}{(n-1)!}x^{-m+2}-\\&-(m-4)!\frac{M_{n-2}}{(n-2)!}x^{-m+3}-(m-5)!\frac{M_{n-3}}{(n-3)!}x^{-m+4}\dots\end{aligned} \tag{23.2}$$

Similarly, a double integration of the initial bimoment gives

$$\begin{aligned}I_2&=(m-1)!\int\int x^{-m}\frac{M_n}{n!}\,dx^2=(m-3)!\frac{M_n}{n!}x^{-m+2}+\\&+(m-4)!\,2\frac{M_{n-1}}{(n-1)!}x^{-m+3}+(m-5)!\,3\frac{M_{n-2}}{(n-2)!}x^{-m+4}+\\&+(m-6)!\,4\frac{M_{n-3}}{(n-3)!}x^{-m+5}+\dots\end{aligned} \tag{23.3}$$

Now we shall apply the derived formulas to a beam whose rigidity is expressed by formula

$$EI_x=\frac{gx^m}{(m-1)!} \tag{23.4}$$

From the differential equation of the flexural axis $EI_x\frac{d^2v}{dx^2}=M_1$ after substitution we obtain

$$\frac{gx^m}{(m-1)!}\cdot\frac{d^2v}{dx^2}=M_1\quad\text{or}\quad g\frac{d^2v}{dx^2}=(m-1)!\,x^{-m}M_1 \tag{23.5}$$

$$\begin{aligned}g\Theta_x&=g\Theta_A+[-M_1u^{-m+1}(m-2)!-M_0u^{-m+2}(m-3)!-\\&-M_{-1}u^{-m+3}(m-4)!-M_{-2}u^{-m+4}(m-5)!]_{u=0}^{u=x}\end{aligned} \tag{23.6}$$

Formula (23.6) is true if $m\geqslant 5$.

If the beam is loaded by force couples, point loads and continuous loads the intensity of which changes according to a linear law, i.e., there are no complex distributed loads (for example parabolic) then in formula (23.6) we obtain a complete series which ends with the term containing the moment M_{-2}.

When substituting the lower limit every load must be taken into account with its initial value which it has at the point where it begins to act on the beam, similarly to the solutions in terms of integrals of the transverse flexibility.

As a result we find

$$\begin{aligned} g\Theta_X = g\Theta_A &- [M_1 x^{-m+1}(m-2)! + M_0 x^{-m+2}(m-3)! + \\ &+ \overline{M}_{-1} x^{-m+3}(m-4)! + M_{-2} x^{-m+4}(m-5)!] + \\ &+ [\sum \overline{M}_i a_i^{-m+1}(m-2)! + \sum P_i b_i^{-m+2}(m-3)! + \\ &+ \sum q_i c_i^{-m+3}(m-4)! + \sum k_i d_i^{-m+4}(m-5)!] \end{aligned} \qquad (23.7)$$

Further integration gives the following equation for deflections

$$\begin{aligned} gv_X = gv_A &+ g\Theta_A(x - x_A) + [M_1 x^{-m+2}(m-3)! + \\ &+ 2M_0 x^{-m+3}(m-4)! + 3M_{-1} x^{-m+4}(m-5)! + 4M_{-2} x^{-m+5}(m-6)!] - \\ &- [\sum \overline{M}_i a_i^{-m+2}(m-3)! + 2\sum P_i b_i^{-m+3}(m-4)! + \\ &+ 3\sum q_i c_i^{-m+4}(m-5)! + 4\sum k_i d_i^{-m+5}(m-6)!] + \\ &+ [(m-2)! \sum \overline{M}_i a_i^{-m+1}(x - a_i) + (m-3)! \sum P_i b_i^{-m+2}(x - b_i) + \\ &+ (m-4)! \sum q_i c_i^{-m+3}(x - c_i) + (m-5)! \sum k_i d_i^{-m+4}(x - d_i)] \end{aligned} \qquad (23.8)$$

If any particular beam is under the influence of a compound load, for example, the one shown in Fig. 22.5, then the expressions for the angular and linear deflections at any point X of such a beam are given by (23.7) and (23.8) after substituting the force factors for this particular section.

24. DETERMINATION OF DISPLACEMENT INTEGRALS BY COEFFICIENTS OF FLEXIBILITY POLYNOMIAL

Formulas of Sec. 23 may be applied to determine the angle Θ_0 between the extreme tangents of the flexural curve and the interval (y_T) between these two tangents measured along any vertical line.

Examine a part of a non-uniform beam AB (Fig. 24.1) in which the rejected parts are represented by forces and force couples. The angular deflection of any section X of such a beam may be determined by (20.3).

Consider the left part AX of the beam

$$\begin{aligned} \Theta_X = \Theta_A &+ \gamma_0 \frac{M_2}{2!} + \gamma_1 \left(x \frac{M_2}{2} - \frac{M_3}{3!} \right) + \\ &+ \gamma_2 \left(\frac{x^2}{2} \cdot \frac{M_2}{2} - x \frac{M_3}{3!} + \frac{M_4}{4!} \right) + \dots \text{ (left forces)} \end{aligned} \qquad (24.1)$$

where x = distance of the section from the origin of coordinates

M_2, M_3, M_4 = moments of corresponding orders of *the left loads* acting on the part AX with respect to the ordinate X-X

Examine the right part BX of the beam. The angle is expressed in terms of moments of the loads to the right of section X; thus

$$\Theta_X = \Theta_B + \Big[-\gamma_0 \frac{M_2}{2!} - \gamma_1 \left(x \frac{M_2}{2!} - \frac{M_3}{3!} \right) - \\ - \gamma_2 \left(\frac{x^2}{2} \cdot \frac{M_2}{2} - x \frac{M_3}{3!} + \frac{M_4}{4!} \right) - \dots \Big] \quad \text{(right forces)} \quad (24.2)$$

where x = distance of the section from the origin, as in the previous formula

M_2, M_3, M_4 = moments of corresponding orders of *the right loads* acting on the part BX, with respect to the ordinate X-X

As the right-hand parts of (24.1) and (24.2) are equal, the angle between the extreme tangents is expressed by

$$\Theta_0 = \Theta_B - \Theta_A = \gamma_0 \frac{M_2}{2} + \gamma_1 \left(x \frac{M_2}{2} - \frac{M_3}{3!} \right) + \\ + \gamma_2 \left(\frac{x^2}{2} \cdot \frac{M_2}{2} - x \frac{M_3}{3!} + \frac{M_4}{4!} \right) + \dots = \text{const} \qquad (24.3)$$

where

$$M_2 = M_2\,(\text{l.f.}) + M_2\,(\text{r.f.}); \qquad M_3 = M_3\,(\text{l.f.}) + M_3\,(\text{r.f.});$$
$$M_4 = M_4\,(\text{l.f.}) + M_4\,(\text{r.f.})$$

are moments of all the loads acting upon the part AB of the beam including the boundary force factors, i.e., force couples and forces applied to the ends of the section AB which replace the rejected parts of the beam.

The angle Θ_0 thus determined is equal to the area of the curvature diagram $\left(\frac{M}{EI}\right)$ evaluated over the part AB; the right-hand side of the latter formula is invariant with respect to the section X, which may be even taken outside the considered interval AB.

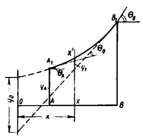

Fig. 24.1

Similarly according to (24.3), the distance between the extreme tangents of the flexural curve A_1B_1 along any vertical line X is given by

$$y=\gamma_0\frac{M_3}{3!}+\gamma_1\left(x\frac{M_3}{3!}-2\frac{M_4}{4!}\right)+\gamma_2\left(\frac{x^2}{2}\cdot\frac{M_3}{3!}-\right.$$
$$\left.-2x\frac{M_4}{4!}+3\frac{M_5}{5!}\right)+\ldots \qquad (24.4)$$

According to the graphical-analytical method, the angle Θ_0 is numerically equal to the total conjugate load

$$\Theta_0=\Omega(k)=M_0(k) \qquad (24.5)$$

i.e., this angle is equal to the area of the flexibility diagram or the zero moment of this diagram. To simplify the calculations, the expressions for the angles in the right-hand part of (24.3) can be written by taking moments with respect to the vertical line passing through the origin of coordinates since this expression is invariant with respect to x

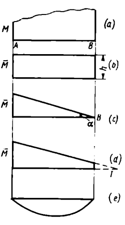

Fig. 24.2

$$\Theta_0=\gamma_0\frac{M_2}{2!}-\gamma_1\frac{M_3}{3!}+\gamma_2\frac{M_4}{4!}-$$
$$-\gamma_3\frac{M_5}{5!}+\ldots=\Omega(k)=M_0(k) \qquad (24.6)$$

Similarly, the interval y of the x vertical line between the extreme tangents

$$y=\gamma_0\frac{M_3}{3!}+\gamma_1\left(x\frac{M_3}{3!}-2\frac{M_4}{4!}\right)+$$
$$+\gamma_2\left(\frac{x^2}{2}\cdot\frac{M_3}{3!}-2x\frac{M_4}{4!}\quad 3\frac{M_5}{5!}\right)+$$
$$+\ldots=S_x(k)=M_1(k) \qquad (24.7)$$

is equal to the static moment or the first moment of the curvature diagram.

As a particular case for the initial vertical line when $x=0$ the interval y_0 is given by

$$y=\gamma_0\frac{M_3}{3!}-2\gamma_1\frac{M_4}{4!}+3\gamma_2\frac{M_5}{5!}-4\gamma_3\frac{M_6}{6!}+\ldots=S_0(k) \qquad (24.8)$$

These formulas may be applied to calculate the displacement $I=\int\frac{M\bar{M}}{EI}dx$ integral for the problem illustrated by Fig. 24.2*a*.

If the bending moment diagram ($\bar{M}$) of the redundant unknown is a rectangle $\bar{M}=h$ (Fig. 24.2*b*), then the displacement integral is given by

$$I=h\int\frac{M}{EI}dx=h\int k\,dx=h\Theta_0 \qquad (24.9)$$

where Θ_0 is determined either by (24.3) or by (24.6).

If the moment diagram of the redundant unknown is an inclined line (Fig. 24.2*c*) then assuming $\overline{M} = pu$, the displacement integral is calculated according to the formula

$$I = \int \frac{M\overline{M}}{EI}\,du = p\int \frac{M}{EI}\,du\cdot u = p\int k\cdot du\cdot u = pS_B(k) = py_B \quad (24.10)$$

where y_B is determined by (24.7) (see Fig. 24.2*e*).

Similarly, for the $\overline{M}$ diagram shown in Fig. 24.2*d*

$$I = py_T$$

A trapezoidal diagram must be replaced by two diagrams, a rectangle and a triangle with the apex at the initial (left) vertical line. Then no additional calculations will be required and the quantities Θ_0 and y_0 determined by (24.6) and (24.8) are sufficient.

Similarly, displacement integrals may be written for parabolic $\overline{M}$ diagrams just like it was previously done for beams of constant rigidity.

25. DETERMINATION OF DISPLACEMENT INTEGRALS BY DERIVATIVES OF FLEXIBILITY ANALYTICAL EXPRESSION

Examine any part AB of a non-uniform beam (Fig. 25.1); in a particular case A and B can be the ends of the beam.

The angle at section X may be determined by considering the left forces according to (21.2)

$$\Theta_X = \Theta_A + \left[\gamma\frac{M_2}{2} - \gamma'\frac{M_3}{3!} + \gamma''\frac{M_4}{4!} - \dots\right] \text{ (left forces)} \quad (25.1)$$

As in Sec. 24, these forces include force couples and forces applied at point A to replace the rejected part of the beam to the *left* of point A.

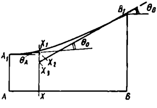

Fig. 25.1

The angle at the same section X may be determined by considering the right forces

$$\Theta_X = \Theta_B + \left[-\gamma\frac{M_2}{2} + \gamma'\frac{M_3}{3!} - \gamma''\frac{M_4}{4!} + \dots\right] \text{ (right forces)} \quad (25.2)$$

This formula contains the moments M_2, M_3, M_4 of forces to the *right* of point X, i.e., within BX.

From (25.1) and (25.2)

$$\Theta_0 = \Theta_B - \Theta_A = \gamma\frac{M_2}{2!} - \gamma'\frac{M_3}{3!} + \dots \quad (25.3)$$

where Θ_0 = angle between the extreme tangents of the flexural curve, which is equal to the area of the curvature diagram or the conjugate load, according to the graphical-analytical method

$M_2 = M_2$ (l.f.) + M_2 (r.f.) and

$M_3 = M_3$ (l.f.) + M_3 (r.f.) are moments of corresponding orders of the right and left forces.

According to Sec. 24 and (25.3), the expression

$$\Theta_0 = \gamma \frac{M_2}{2!} - \gamma' \frac{M_3}{3!} + \gamma'' \frac{M_4}{4!} - \ldots = \Omega(k) \qquad (25.4)$$

does not depend on the position of the moment point.

In (25.4) moments of all the forces, left and right, may be taken about an arbitrary point X. Then for any balanced force system, after reducing similar terms, the variable x vanishes.

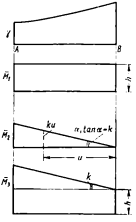

Fig. 25.2

Similarly, the distance $\overline{X_2X_3}$ between the extreme tangents of the flexural curve measured along the x vertical line is numerically equal to the static moment relative to the vertical of the conjugate load acting upon the part AB; it can be calculated from the formula

$$\overline{X_2X_3} = S_x(\Omega_k) = \gamma \frac{M_3}{3!} - $$

$$- 2\gamma' \frac{M_4}{4!} + 3\gamma'' \frac{M_5}{5!} \ \ldots \qquad (25.5)$$

Finally, the area of figure $A_1X_1B_1X_3X_2A_1$ outlined by the flexural curve and the boundary tangents is numerically equal to the second moment of the curvature diagram with respect to the x vertical

$$\gamma \frac{M_4}{4!} - 3\gamma' \frac{M_5}{5!} + 6\gamma'' \frac{M_6}{6!} \ \ldots \qquad (25.6)$$

These formulas may be applied to calculate displacement integrals of the following type

$$\int \frac{M_p \bar{M}}{EI} dx$$

Suppose the flexibility γ changes along the beam according to an algebraic polynomial or any other law (Fig. 25.2); let such a beam be loaded by an arbitrary load, and the bending moment diagram of the redundant unknown is represented by a rectangle $\overline{M}_1$, a triangle $\overline{M}_2$ or a trapezium $\overline{M}_3$.

For a rectangle the integral

$$I_1 = \int \frac{\overline{M} M_p}{EI}\,dx = \overline{M}_1 \int \frac{M_p}{EI}\,dx$$

may be expressed according to (25.4) by

$$I_1 = \overline{M}_1 \left[\gamma \frac{M_2}{2!} - \gamma' \frac{M_3}{3!} + \gamma'' \frac{M_4}{4!} - \dots \right] \tag{25.7}$$

The higher moments in the brackets are determined from formula (I) of Sec. 6, where the moment point may be chosen arbitrarily.

For the triangular diagram $\overline{M}_2$ of the bending moment the integral is

$$I_2 = \int \frac{\overline{M}_2 M_p}{EI}\,du = k \int u \frac{M_p}{EI}\,du \tag{25.8}$$

Since I_2 is a static moment of the conjugate load (i.e., loading of a beam with a curvature diagram), then according to formula (25.5) it may be expressed by

$$I_2 = k \left[\gamma \frac{M_3}{3!} - 2\gamma' \frac{M_4}{4!} + 3\gamma'' \frac{M_5}{5!} - \dots \right] \tag{25.9}$$

where the higher moments are calculated according to formula (I) and a section with a zero ordinate in the $\overline{M}_2$ diagram must be taken as the moment point.

Finally, if the diagram of the redundant unknown is a trapezium $\overline{M}_3$, then it may be replaced by two diagrams—a rectangle and a triangle; in that way the problem is reduced to the two previous cases. Thus, it is sufficient to determine I_1 and I_2 according to formulas (25.7) and (25.9) in order to solve any linear diagrams of redundant unknowns.

If the moment diagram of a redundant unknown is a parabola then expression (25.6) must be used to determine the integral.

Chapter V

Multispan Non-Uniform Beams

26. EQUATION OF THREE MOMENTS IN FLEXIBILITY POLYNOMIAL COEFFICIENTS

Let us take a continuous beam consisting of several spans of continuously varying cross sections, their rigidity being expressed by algebraic polynomials.

Any intermediate span may be considered as a separate beam on two supports under the influence of two positive force couples at its ends (Fig. 26.1). These force couples replace the rejected spans of the beam.

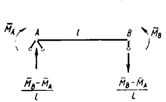
$\bar{M}_A$
A
l
B
$\bar{M}_B$
$\frac{\bar{M}_B-\bar{M}_A}{l}$
$\frac{\bar{M}_B-\bar{M}_A}{l}$

Fig. 26.1

The n-th moment of all the loads (divided by $n!$) with respect to the right support is given by

$$\frac{M_{B^n}}{n!}=\bar{M}_A\frac{l^{n-1}}{(n-1)!}+\frac{\bar{M}_B-\bar{M}_A}{l}\cdot\frac{l^n}{n!}=$$

$$=\frac{1}{l}[\bar{M}_A(n-1)+\bar{M}_B]\frac{l^n}{n!} \tag{26.1}$$

Particular values for the n-th moment are derived from this formula by substituting $n=2,\ 3,\ 4\ldots$

According to the equation of angular displacements of non-uniform beams (20.3), the following expressions are required in calculations

$$\left.\begin{aligned}\left[x\frac{M_2}{2}-\frac{M_3}{3!}\right]_{x=l}&=\frac{1}{l}[\overline{M}_A+2\overline{M}_B]\frac{l^3}{3!}\\ \left[\frac{x^2}{2}\cdot\frac{M_2}{2}-x\frac{M_3}{3!}+\frac{M_4}{4!}\right]_{x=l}&=\frac{1}{l}[\overline{M}_A+3\overline{M}_B]\frac{l^4}{4!}\end{aligned}\right\}\tag{26.2}$$

The regularity observed in these two equalities makes it possible to compose further expressions without calculations

$$\left.\begin{aligned}&\left[\frac{x^3}{3!}\cdot\frac{M_2}{2}-\frac{x^2}{2}\cdot\frac{M_3}{3!}+x\frac{M_4}{4!}-\frac{M_5}{5!}\right]_{x=l}=\\ &\qquad=\frac{1}{l}[\overline{M}_A+4\overline{M}_B]\frac{l^5}{5!}\\ &\left[\frac{x^4}{4!}\cdot\frac{M_2}{2}-\frac{x^3}{3!}\cdot\frac{M_3}{3!}+\frac{x^2}{2}\cdot\frac{M_4}{4!}-\right.\\ &\left.-x\frac{M_5}{5!}+\frac{M_6}{6!}\right]_{x=l}=\frac{1}{l}[\overline{M}_A+5\overline{M}_B]\frac{l_6}{6!},\ \text{etc.}\end{aligned}\right\}\tag{26.3}$$

Similarly,

$$\left.\begin{aligned}&\left[x\frac{M_3}{3!}-2\frac{M_4}{4!}\right]_{x=l}=l\frac{1}{l}(2\overline{M}_A+\overline{M}_B)\frac{l^3}{3!}-\\ &\quad-2\frac{1}{l}(2\overline{M}_A+\overline{M}_B)\frac{l^4}{4!}=\frac{1}{l}[2\overline{M}_A+2\overline{M}_B]\frac{l^4}{4!}\\ &\qquad\left[\frac{x^2}{2}\cdot\frac{M_3}{3!}-2x\frac{M_4}{4!}+3\frac{M_5}{5!}\right]_{x=l}=\\ &=\frac{l^2}{2}\cdot\frac{1}{l}(2\overline{M}_A+\overline{M}_B)\frac{l^3}{3!}-2l\frac{1}{l}(3\overline{M}_A+\overline{M}_B)\frac{l^4}{4!}+\\ &\quad+3\frac{1}{l}(4\overline{M}_A+\overline{M}_B)\frac{l^5}{5!}=\frac{1}{l}(2\overline{M}_A+3\overline{M}_B)\frac{l^5}{5!},\ \text{etc.}\end{aligned}\right\}\tag{26.4}$$

Other binomials are similarly composed.

According to the arrangement of supports, the boundary conditions of this particular span are expressed by

$$v_A=0,\quad \Theta_A\neq 0$$

The angle Θ_A is determined from the equation when $v_B=0$. From (20.3) we obtain

$$\begin{aligned}\Theta_A=&-\frac{\overline{M}_B}{l^2}\left(\gamma_0\frac{l^3}{3!}+\gamma_1\cdot 2\frac{l^4}{4!}+\gamma_2\cdot 3\frac{l^5}{5!}+\gamma_3\cdot 4\frac{l^6}{6!}+\dots\right)-\\ &-2\frac{\overline{M}_A}{l^2}\left(\gamma_0\frac{l^3}{3!}+\gamma_1\frac{l^4}{4!}+\gamma_2\frac{l^5}{5!}+\gamma_3\frac{l^6}{6!}+\dots\right)\end{aligned}\tag{26.5}$$

A definite regularity may be observed in the sequence and in the structure of terms in this polynomial.

The angle Θ_A determined for the left support is a result of loading the beam AB by force couples M_A and M_B at its ends with no span loading.

If $\gamma = \gamma_0$ ($\gamma_1 = \gamma_2 = \gamma_3 = \ldots = 0$) then a well-known formula for uniform beams is derived from (26.5) by substituting

$$\Theta_A = -\frac{\overline{M}_B l}{6EI} - \frac{\overline{M}_A l}{3EI}$$

According to (20.3), the angle Θ_B at the right support of a non-uniform beam is

$$\Theta_B = \frac{\overline{M}_B}{l^2}\left(\gamma_0 \cdot 1 \cdot 2\frac{l^3}{3!} + \gamma_1 \cdot 2 \cdot 3\frac{l^4}{4!} + \gamma_2 \cdot 3 \cdot 4\frac{l^5}{5!} + \right.$$
$$\left. + \gamma_3 \cdot 4 \cdot 5\frac{l^6}{6!} + \ldots\right) + \frac{\overline{M}_A}{l^2}\left(\gamma_0\frac{l^3}{3!} + \gamma_1 \cdot 2 \cdot \frac{l^4}{4!} + \right.$$
$$\left. + \gamma_2 \cdot 3\frac{l^5}{5!} + \gamma_3 \cdot 4 \cdot \frac{l^6}{6!} + \ldots\right) \tag{26.6}$$

Comparison of expressions obtained for angles Θ_A and Θ_B leads to the following conclusions. Each expression consists of two terms, one of which presents the angle caused by moment $\overline{M}_A$ and the other, by $\overline{M}_B$; at the same time the angle at support A caused by the couple $\overline{M}_B$ is equal to the angle at support B caused by the couple $\overline{M}_A$. This follows from the complete identity of the expressions

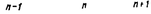

Fig. 26.2

$$\frac{1}{l^2}\left(\gamma_0\frac{l^3}{3!} + \gamma_1 \cdot 2\frac{l^4}{4!} + \gamma_2 \cdot 3\frac{l^5}{5!} + \gamma_3 \cdot 4\frac{l^6}{6!} + \ldots\right)$$

An equation of three moments will now be composed for a multispan beam of continuously varying rigidity by examining two adjacent spans with supports $(n - 1)$, n and $(n + 1)$ (Fig. 26.2).

The angle at the middle support n may be determined in two ways: first, by considering loads to the left of support n and, second, considering the loads to the right of support n. These two expressions must be equal, i.e.

$$\Theta_{n,\,n} + \Theta^0_{n,\,n} = \Theta_{n,\,n+1} + \Theta^0_{n,\,n+1}$$

where $\Theta_{n,\,n}$ = angle at support n caused by support moments $\overline{M}_{n-1}$ and $\overline{M}_n$ loading span n

$\Theta^0_{n,\,n}$ = same angle caused by loads on span n if this span is considered a separate simple beam

$\Theta_{n,\,n+1}$ = same angle caused by support moments $\overline{M}_n$ and $\overline{M}_{n+1}$ loading span $(n+1)$

$\Theta^0_{n,\,n+1}$ = same angle caused by loads on span $(n+1)$ if this span is considered a separate simple beam.

The angle $\Theta_{n,\,n}$ is determined from the formula derived for Θ_B if we assume $\overline{M}_A = \overline{M}_{n-1}$ and $\overline{M}_B = \overline{M}_n$; and for the angle $\Theta_{n,\,n+1}$ we assume $\overline{M}_A$ for $\overline{M}_n$ and $\overline{M}_B$ for $\overline{M}_{n+1}$. Substituting these expressions into the previous equations (26.5) and (26.6) and transferring terms $\Theta^0_{n,\,n}$ and $\Theta^0_{n,\,n+1}$ into the right-hand side of the equation we obtain

$$\frac{\overline{M}_{n-1}}{l_n^2}\left(\gamma_{0,\,n}\frac{l_n^3}{3!}+\gamma_{1,\,n}\cdot 2\,\frac{l_n^4}{4!}+\gamma_{2,\,n}\cdot 3\,\frac{l_n^5}{5!}+\gamma_{3,\,n}\cdot 4\,\frac{l_n^6}{6!}+\dots\right)+$$
$$+\frac{\overline{M}_n}{l_n^2}\left(\gamma_{0,\,n}\cdot 1\cdot 2\frac{l_n^3}{3!}+\gamma_{1,\,n}\cdot 2\cdot 3\,\frac{l_n^4}{4!}+\gamma_{2,\,n}\cdot 3\cdot 4\cdot\frac{l_n^5}{5!}+\right.$$
$$\left.+\gamma_{3,\,n}\cdot 4\cdot 5\,\frac{l_n^6}{6!}+\dots\right)+\frac{\overline{M}_n}{l_{n+1}^2}\,2\left(\gamma_{0,\,n+1}\frac{l_{n+1}^3}{3!}+\gamma_{1,\,n+1}\frac{l_{n+1}^4}{4!}+\right.$$
$$\left.+\gamma_{2,\,n+1}\frac{l_{n+1}^5}{5!}+\gamma_{3,\,n+1}\frac{l_{n+1}^6}{6!}+\dots\right)+\frac{\overline{M}_{n+1}}{l_{n+1}^2}\left(\gamma_{0,\,n+1}\frac{l_{n+1}^3}{3!}+\right.$$
$$+\gamma_{1,\,n+1}\cdot 2\frac{l_{n+1}^4}{4!}+\gamma_{2,\,n+1}\cdot 3\frac{l_{n+1}^5}{5!}+$$
$$\left.+\gamma_{3,\,n+1}\cdot 4\,\frac{l_{n+1}^6}{6!}+\dots\right)=\Theta^0_{n,\,n+1}-\Theta^0_{n,\,n} \qquad (26.7)$$

To simplify the calculations of angles $\Theta^0_{n,\,n+1}$ and $\Theta^0_{n,\,n}$, caused by span loads when the spans are considered simple beams, the formulas for the most popular types of loading may be composed.

For example, determine angles Θ_A and Θ_B for a uniform load of intensity q distributed along the entire beam length l (Fig. 26.3).

Since $v_B = 0$ when $x = l$, formula (20.4) becomes

$$\Theta_A l+\gamma_0\frac{M_3}{3!}+\gamma_1\left(l\,\frac{M_3}{3!}-2\,\frac{M_4}{4!}\right)+$$
$$+\gamma_2\left(\frac{l^2}{2}\cdot\frac{M_3}{3!}-2l\,\frac{M_4}{4!}+3\,\frac{M_5}{5!}\right)+\dots=0$$

Expressions in parentheses are composed as before

$$\frac{M_3}{3!}=\frac{ql}{2}\cdot\frac{l^3}{6}-\frac{ql^4}{24}=\frac{ql^4}{4!}$$

and

$$l\frac{M_3}{3!}-2\frac{M_4}{4!}=l\frac{ql^4}{24}-2\left(\frac{ql}{2}\ \frac{l^4}{24}-\frac{ql^5}{120}\right)=\frac{ql^5}{60}=2\frac{ql^5}{5!}$$

an so on, by analogy

$$\frac{l^2}{2}\cdot\frac{M_3}{3!}-2l\frac{M_4}{4!}+3\frac{M_5}{5!}=3\frac{ql^6}{6!}\text{, etc.}$$

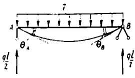

Fig. 26.3

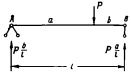

Fig. 26.4

Substituting these expressions into the equation and solving the latter for Θ_A we find that

$$\Theta_A=-\frac{1}{l}\left(\gamma_0\frac{ql^4}{4!}+\gamma_1\cdot 2\frac{ql^5}{5!}+\gamma_2\cdot 3\frac{ql^6}{6!}+\gamma_3\cdot 4\frac{ql^7}{7!}+\dots\right)\qquad(26.8)$$

Knowing Θ_A and using formula (20.3) we find the angle

$$\Theta_B=\Theta_A+\gamma_0\frac{M_2}{2!}+\gamma_1\left(l\frac{M_2}{2!}-\frac{M_3}{3!}\right)+$$

$$+\gamma_2\left(\frac{l^2}{2!}\cdot\frac{M_2}{2!}-l\frac{M_3}{3!}+\frac{M_4}{4!}\right)+\dots$$

Substitution of (26.8) into this formula gives

$$\Theta_B=\frac{1}{2l}\left(\gamma_0\cdot 1\cdot 2\frac{ql^4}{4!}+\gamma_1\cdot 2\cdot 3\frac{ql^5}{5!}+\right.$$

$$\left.+\gamma_2\cdot 3\cdot 4\frac{ql^6}{6!}+\gamma_3\cdot 4\cdot 5\frac{ql^7}{7!}+\dots\right)\qquad(26.9)$$

For a point load P (Fig. 26.4)

$$\Theta_A=-\frac{2Pa}{l}\left[\frac{\gamma_0}{3!}(l^2-a^2)+\frac{\gamma_1}{4!}(l^3-a^3)+\right.$$

$$\left.+\frac{\gamma_2}{5!}(l^4-a^4)+\frac{\gamma_3}{6!}(l^5-a^5)+\dots\right]+$$

$$+\frac{Pb}{l}\left(\gamma_0\frac{a^2}{2}+\gamma_1\frac{a^3}{3!}+\gamma_2\frac{a^4}{4!}+\dots\right)\qquad(26.10)$$

$$\Theta_B=\frac{Pa}{l}\left[\frac{\gamma_0}{3!}(l^2-a^2)+\frac{2\gamma_1}{4!}(l^3-a^3)+\right.$$

$$\left.+\frac{3\gamma_2}{5!}(l^4-a^4)+\frac{4\gamma_3}{6!}(l^5-a^5)+\dots\right]\qquad(26.11)$$

By generalizing all the terms of (26.11) when $a = b = \frac{l}{2}$ we obtain

$$\Theta_B = \sum_{i=0,\,1,\,2\,\dots} \frac{\gamma_i}{l} P \frac{l^{i+3}}{(i+3)!} \cdot \frac{i+1}{2} \tag{26.12}$$

27. EQUATION OF THREE MOMENTS IN DERIVATIVES OF ANALYTICALLY EXPRESSED FLEXIBILITY

Formulas presented in the previous section are based on the first method of designing non-uniform beams, which deals with flexibility polynomial coefficients. These formulas may be applied to derive a general solution for multispan beams, but such an approach requires quite cumbersome preliminary calculations. Besides, these formulas may be applied only if the flexibility of a beam is expressed by an algebraic polynomial.

An equation of three moments will now be derived by the second method of designing non-uniform beams (Sec. 21), in which the flexibility of a beam can be expressed by any analytical law.

Consider the basic formulas (21.2) and (21.3), in which $\gamma', \gamma'', \gamma''' \dots$ are the first, second, third, etc. derivatives of γ with respect to x, and M_2, M_3, M_4, ... are moments of the second, third, fourth, etc. order of all forces (both active and reactive) to the left of section X relative to the centre of gravity of this section.

The following general formula will be applied to determine higher moments for the load shown in Fig. 22.2

$$\frac{M_n}{n!} = EIv_A \frac{x^{n-3}}{(n-3)!} + EI\Theta_A \frac{x^{n-2}}{(n-2)!} + \sum \overline{M}_i \frac{(x-a_i)^{n-1}}{(n-1)!} + \\ + \sum P_i \frac{(x-b_i)^n}{n!} + \sum q_i \frac{(x-c_i)^{n+1}}{(n+1)!} + \sum k_i \frac{(x-d_i)^{n+2}}{(n+2)!} \tag{27.1}$$

Substituting $n = 0, 1, 2, 3, \dots$ into (27.1), we arrive at expressions for the moment of zero order (transverse force at section X with abscissa x), the first moment (bending moment), second, third, etc. moments, divided by the respective factorials corresponding to the orders of the moments.

Angular and linear displacements for any section of the beam are determined from (21.2) and (21.3), the initial values of Θ_A and v_A being found from the boundary conditions.

For example, let a simple beam of non-uniform section (see Fig. 26.1) be loaded by force couples $\overline{M}_A$ and $\overline{M}_B$ at supports; since the deflection at the right support is zero, applying formula (21.3) we find

$$v_B = \Theta_A l + \frac{\overline{M}_B - \overline{M}_A}{l}\left[\gamma_B \frac{l^3}{3!} - 2\gamma'_B \frac{l^4}{4!} + \dots\right] + \\ + \overline{M}_A \left[\gamma_B \frac{l^2}{2} - 2\gamma'_B \frac{l^3}{3!} + \dots\right] = 0$$

hence,

$$\Theta_A = -\frac{\overline{M}_B}{l^2}\left[\gamma_B\frac{l^3}{3!} - 2\gamma'_B\frac{l^4}{4!} + 3\gamma''_B\frac{l^5}{5!} - 4\gamma'''_B\frac{l^6}{6!}\ \dots\right] -$$
$$-\frac{\overline{M}_A}{l^2}\left[1\cdot 2\gamma_B\frac{l^3}{3!} - 2\cdot 3\gamma'_B\frac{l^4}{4!} + 3\cdot 4\gamma''_B\frac{l^5}{5!} - 4\cdot 5\gamma'''_B\frac{l^6}{6!} + \dots\right] \quad (27.2)$$

Substituting this expression into (21.2) and assuming $x = l$

$$\Theta_B = 2\frac{\overline{M}_B}{l^2}\left[\gamma_B\frac{l^3}{3!} - \gamma'_B\frac{l^4}{4!} - \gamma''_B\frac{l^5}{5!} - \dots\right] +$$
$$+\frac{\overline{M}_A}{l^2}\left[\gamma_B\frac{l^3}{3!} - 2\gamma''_B\frac{l^4}{4!} + 3\gamma''_B\frac{l^5}{5!} - 4\gamma'''_B\frac{l^6}{6!} + \dots\right] \quad (27.3)$$

Equating the angles to the left and to the right of the middle support n and using formulas (27.2) and (27.3), an equation of three moments is obtained for a non-uniform multispan beam, as in the previous section:

$$\frac{\overline{M}_{n-1}}{l_n^2}\left[\gamma_{n,n}\frac{l_n^3}{3!} - 2\gamma'_{n,n}\frac{l_n^4}{4!} + 3\gamma''_{n,n}\frac{l_n^5}{5!} - 4\gamma'''_{n,n}\frac{l_n^6}{6!} + \dots\right] +$$
$$+2\frac{\overline{M}_n}{l_n^2}\left[\gamma_{n,n}\frac{l_n^3}{3!} - \gamma'_{n,n}\frac{l_n^4}{4!} + \gamma''_{n,n}\frac{l_n^5}{5!} - \gamma'''_{n,n}\frac{l_n^6}{6!} + \dots\right] +$$
$$+2\frac{\overline{M}_n}{l_{n+1}^2}\left[\gamma_{n+1,n+1}\frac{l_{n+1}^3}{3!} - 3\gamma'_{n+1,n+1}\frac{l_{n+1}^4}{4!} +\right.$$
$$\left.+6\gamma''_{n+1,n+1}\frac{l_{n+1}^5}{5!} - 10\gamma'''_{n+1,n+1}\frac{l_{n+1}^6}{6!} + \dots\right] +$$
$$+\frac{\overline{M}_{n+1}}{l_{n+1}^2}\left[\gamma_{n+1,n+1}\frac{l_{n+1}^3}{3!} - 2\gamma'_{n+1,n+1}\frac{l_{n+1}^4}{4!} +\right.$$
$$\left.+3\gamma''_{n+1,n+1}\frac{l_{n+1}^5}{5!} - 4\gamma'''_{n+1,n+1}\frac{l_{n+1}^6}{6!} + \dots\right] = \Theta^0_{n,n+1} - \Theta^0_{n,n} \quad (27.4)$$

where

$\Theta^0_{n,n+1} - \Theta^0_{n,n}$ = the difference of angles at the n-th support for the $(n+1)$-th and n-th spans if they are considered as simple beams
$\overline{M}_{n-1}, \overline{M}_n, \overline{M}_{n+1}$ = moments at supports $(n-1)$, n and $(n+1)$, respectively
$\gamma_{n,n}\ \gamma'_{n,n}, \gamma''_{n,n}$ = flexibility and its derivatives at the n-th support of span n, i.e., at the end of the n-th span
$\gamma_{n+1,n+1}, \gamma'_{n+1,n+1}, \gamma''_{n+1,n+1}$ = flexibility and its derivatives at the $(n+1)$-th support of span $(n+1)$, i.e., at the end of the $(n+1)$-th span

It is advisable to compose the formulas for angles at supports of a simple beam beforehand for the most popular types of loadings in order to simplify the solution of the three moment system of equations.

For example, if a beam is loaded over the entire span with a uniform load (Fig. 26.3), then from formulas (21.2) and (21.3) we obtain

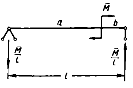

Fig. 27.1

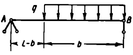

Fig. 27.2

$$\Theta_A = -\frac{q}{l}\left[\gamma_B \frac{l^4}{4!} - 3\gamma_B' \frac{l^5}{5!} + 6\gamma_B'' \frac{l^6}{6!} - 10\gamma_B''' \frac{l^7}{7!} + \dots\right] \quad (27.5)$$

$$\Theta_B = \frac{q}{l}\left[\gamma_B \frac{l^4}{4!} - 2\gamma_B' \frac{l^5}{5!} + 3\gamma_B'' \frac{l^6}{6!} - 4\gamma_B''' \frac{l^7}{7!} + \dots\right] \quad (27.6)$$

For a single point load (Fig. 26.4)

$$\Theta_A = -\frac{Pb}{l}\left[\gamma_B \frac{l^2-b^2}{3!} - 2\gamma_B' \frac{l^3-b^3}{4!} + \right.$$
$$\left. + 3\gamma_B'' \frac{l^4-b^4}{5!} - 4\gamma_B''' \frac{l^5-b^5}{6!} + \dots\right] \quad (27.7)$$

$$\Theta_B = \frac{Pb}{l}\left\{\frac{\gamma_B}{3!}[3l(l-b)-(l^2-b^2)] - \frac{\gamma_B'}{4!}[4l(l^2-b^2) - \right.$$
$$-2(l^3-b^3)] + \frac{\gamma_B''}{5!}[5l(l^3-b^3)-3(l^4-b^4)] -$$
$$\left. -\frac{\gamma_B'''}{6!}[6l(l^4-b^4)-4(l^5-b^5)] + \dots\right\} \quad (27.8)$$

For a force couple (Fig. 27.1)

$$\Theta_A = \frac{\bar{M}}{l}\left[\gamma_B\left(\frac{l^2}{3!}-\frac{b^2}{2}\right) - 2\gamma_B'\left(\frac{l^3}{4!}-\frac{b^3}{3!}\right) + \right.$$
$$\left. + 3\gamma_B''\left(\frac{l^4}{5!}-\frac{b^4}{4!}\right) - 4\gamma_B'''\left(\frac{l^5}{6!}-\frac{b^5}{5!}\right) + \dots\right] \quad (27.9)$$

$$\Theta_B = \frac{\bar{M}}{l}\left[\gamma_B\left(-\frac{2}{3!}l^2 + \frac{bl}{1!} - \frac{1b^2}{2!}\right) - \gamma_B'\left(-\frac{2}{4!}l^3 + \frac{b^2l}{2!} - \frac{2}{3!}b^3\right) + \right.$$
$$\left. + \gamma_B''\left(-\frac{2}{5!}l^4 + \frac{b^3l}{3!} - \frac{3}{4!}b^4\right) - \gamma_B'''\left(-\frac{2}{6!}l^5 + \frac{b^4l}{4!} - \frac{4}{5!}b^5\right) + \dots\right]$$
$$(27.10)$$

For a uniform load over a part of the span (Fig. 27.2)

$$\Theta_A = -\frac{q}{l}\left[\gamma_B\left(\frac{b^2}{2}\cdot\frac{l^2}{3!}-\frac{b^4}{4!}\right)-2\gamma_B'\left(\frac{b^2}{2}\cdot\frac{l^3}{4!}-\frac{b^5}{5!}\right)+\right.$$
$$\left.+3\gamma_B''\left(\frac{b^2}{2}\cdot\frac{l^4}{5!}-\frac{b^6}{6!}\right)-4\gamma_B'''\left(\frac{b^2}{2}\cdot\frac{l^5}{6!}-\frac{b^7}{7!}\right)+\dots\right] \qquad (27.11)$$

$$\Theta_B = \frac{qb^2}{l}\left\{\gamma_B\left[\frac{b^2}{4}+\frac{l(l-b)}{3!}\right]-\gamma_B'\left[\frac{2b^3}{5!}+\frac{l(l^2-b^2)}{4!}\right]+\right.$$
$$\left.+\gamma_B''\left[\frac{3b^4}{6!}+\frac{l(l^3-b^3)}{5!}\right]-\gamma_B'''\left[\frac{4b^5}{7!}+\frac{l(l^4-b^4)}{6!}\right]+\dots\right\} \qquad (27.12)$$

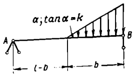

Fig. 27.3

Finally, for a triangular load acting upon a part of the span (Fig. 27.3)

$$\Theta_A = -\frac{k}{l}\left[\gamma_B\left(\frac{b^3}{3!}\cdot\frac{l^2}{3!}-\frac{b^5}{5!}\right)-2\gamma_B'\left(\frac{b^3}{3!}\cdot\frac{l^3}{4!}-\frac{b^6}{6!}\right)+\right.$$
$$\left.+3\gamma_B''\left(\frac{b^3}{3!}\cdot\frac{l^4}{5!}-\frac{b^7}{7!}\right)-4\gamma_B'''\left(\frac{b^3}{3!}\cdot\frac{l^5}{6!}-\frac{b^8}{8!}\right)+\dots\right] \qquad (27.13)$$

$$\Theta_B = \frac{k}{l}\left[\gamma_B\left(-2\frac{b^3}{3!}\cdot\frac{l^2}{3!}+\frac{b^4 l}{4!}-\frac{b^5}{5!}\right)-\gamma_B'\left(-2\frac{b^3}{3!}\cdot\frac{l^3}{4!}+\right.\right.$$
$$\left.+\frac{b^5 l}{5!}-2\frac{b^6}{6!}\right)+\gamma_B''\left(-2\frac{b^3}{3!}\cdot\frac{l^4}{5!}+\frac{b^6 l}{6!}-3\frac{b^7}{7!}\right)-$$
$$\left.-\gamma_B'''\left(-2\frac{b^3}{3!}\cdot\frac{l^5}{6!}+\frac{b^7 l}{7!}-4\frac{b^8}{8!}\right)+\dots\right] \qquad (27.14)$$

More compound loads may be replaced by a sum of the loads examined above.

28. EQUATION OF THREE MOMENTS IN FLEXIBILITY INTEGRALS

The third method of evaluating non-uniform beams outlined in Sec. 22 may be applied to derive an equation of three moments for a multispan non-uniform beam in terms of flexibility integrals for the separate spans.

Formulas (22.3) and (22.6) determine angular and linear deflections

$$\begin{aligned}\Theta_X = {} & \Theta_A + (M_X\Gamma_{1,x} - Q_X\Gamma_{2,x} + q_X\Gamma_{3,x} - k_X\Gamma_{4,x}) - \\ & - [\sum \overline{M}_i\Gamma_{1,a_i} - \sum P_i\Gamma_{2,b_i} + \sum q_i\Gamma_{3,c_i} - \sum k_i\Gamma_{4,d_i}]\end{aligned} \qquad (28.1)$$

$$\begin{aligned}v_X = {} & [v_A + \Theta_A x + M_X\Gamma_{2,x} - 2Q_X\Gamma_{3,x} + 3q_X\Gamma_{4,x} - 4k_X\Gamma_{5,x}] - \\ & - [\sum \overline{M}_i\Gamma_{2,a_i} - 2\sum P_i\Gamma_{3,b_i} + 3\sum q_i\Gamma_{4,c_i} - 4\sum k_i\Gamma_{5,d_i}] - \\ & - [\sum M_i\Gamma_{1,a_i}(x-a_i) - \sum P_i\Gamma_{2,b_i}(x-b_i) + \\ & + \sum q_i\Gamma_{3,c_i}(x-c_i) - \sum k_i\Gamma_{4,d_i}(x-d_i)]\end{aligned} \qquad (28.2)$$

These formulas are quite simple to apply, although they look very complicated. They allow for any possible combination of loadings.

However, for particular types of loads used in engineering practice most of the terms vanish from formulas (28.1) and (28.2), as well as from the equation of three moments.

It will be recalled that Γ_1, Γ_2, Γ_3, Γ_4 and Γ_5 are successive integrals of the flexibility, i.e., $\Gamma_n = \underbrace{\int \ldots \int}_{n\ \text{times}} \gamma dx^n$; figures in the double subscripts of Γ denote the multiplicity of the integral, while the letters give the right boundary point of the abscissa, i.e., the upper integration limit (the lower limit corresponds to the left-hand end of the beam, i.e., to the zero abscissa). For example,

$$\Gamma_{1,x} = \int\limits_{u=0}^{u=x} \gamma du; \quad \Gamma_{2,b_i} = \iint\limits_{u=0}^{u=b_i} \gamma du^2; \quad \Gamma_{3,c_i} = \iiint\limits_{u=0}^{u=c_i} \gamma du^3, \text{ etc.} \qquad (28.3)$$

In formulas (28.1) and (28.2)

M_X = bending moment at section X with abscissa x (Fig. 22.2)
Q_X = transverse force at the same section
q_X = intensity of uniform load
k_X = derivative of intensity (slope of triangular loaded area with its apex pointing to the left)
M_i = force couples ($i = 0, 1, 2, 3 \ldots$) (Fig. 28.1)
P_i = point loads ($i = 0, 1, 2, 3 \ldots$)
q_i = intensities of uniform loads ($i = 0, 1, 2 \ldots$)
k_i = slopes of uniformly increasing (in absolute value) loads represented by triangular loaded areas with apexes pointing to the left ($i = 0, 1, 2, 3 \ldots$)

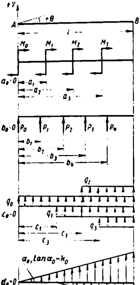

Fig. 28.1

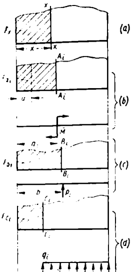

Fig. 28.2

Formulas (22.17), (22.18) and (22.19) elucidate the meaning of flexibility integrals (Γ); according to these formulas, flexibility integrals may be replaced by higher moments of the left parts of flexibility diagrams (Fig. 28.2). Quantities included in formulas (28.1), (28.2), (22.18) and (22.19) may be now interpreted graphically

$$\Gamma_{1,x} = M_{X^0} = F_X; \quad \Gamma_{2,x} = M_{X^1}; \quad \Gamma_{3,x} = \frac{M_{X^2}}{2};$$

$$\Gamma_{4,x} = \frac{M_{X^3}}{3!}; \quad \Gamma_{5,x} = \frac{M_{X^4}}{4!} \tag{28.4}$$

where F_X = part of the flexibility diagram between $u = 0$ and $u = x$ (Fig. 28.2*a*)

M_{X^1}, M_{X^2}, M_{X^3}, M_{X^4} = first, second, third and fourth moments of area F_X with respect to axis X-X.

Similarly,

$$\Gamma_{1,a_i} = F_{a_i}; \quad \Gamma_{2,a_i} = M_{A_i^1} \tag{28.5}$$

where F_{a_i} = part of the flexibility diagram between $u = 0$ and $u = a_i$, i.e., lying to the left of the point of application of the couple $\overline{M}_i$

$M_{A_i^1}$ = first (static) moment of area F_{a_i} with respect to vertical line A_i-A_i (Fig. 28.2*b*)

Similarly,

$$\Gamma_{2,b_i} = M_{B_i^1}; \quad \Gamma_{3,b_i} = \frac{M_{B_i^2}}{2} \tag{28.6}$$

where $M_{B_i^1}$, $M_{B_i^2}$ = first and second moments of the left part of the flexibility diagram with respect to axis B_i-B_i (Fig. 28.2*c*)

And

$$\Gamma_{3,c_i} = \frac{M_{C_i^2}}{2}; \quad \Gamma_{4,c_i} = \frac{M_{C_i^3}}{3!} \tag{28.7}$$

where $M_{C_i^2}$, $M_{C_i^3}$ = second and third moments of the left part of the flexibility diagram with respect to axis C_i-C_i (Fig. 28.2*d*)

Finally,

$$\Gamma_{4,d_i} = \frac{M_{D_i^3}}{3!}; \quad \Gamma_{5,d_i} = \frac{M_{D_i^4}}{4!} \tag{28.8}$$

where $M_{D_i^3}$, $M_{D_i^4}$ = third and fourth moments of the left part of the flexibility diagram with respect to axis D_i-D_i (Fig. 28.2*e*)

In order to derive the three-moment equation, the angles at supports A and B of a simple beam will be determined for the force couples $\overline{M}_A$ and $\overline{M}_B$ applied to the ends (Fig. 28.3).

For this particular case when $x=l$ from formula (22.6) we obtain

$$v_B=\Theta_A l+\overline{M}_B\Gamma_{2,\,l}-2\frac{M_B-M_A}{l}\Gamma_{3,\,l}=0$$

hence,

$$\Theta_A=-\overline{M}_B l\left(\frac{\Gamma_{2,\,l}}{l^2}-2\frac{\Gamma_{3,\,l}}{l^3}\right)-\overline{M}_A l\frac{2\Gamma_{3,\,l}}{l^3} \tag{28.9}$$

From formula (22.3)

$$\Theta_B=\Theta_A+\overline{M}_B\Gamma_{1,\,l}+\frac{\overline{M}_B-\overline{M}_A}{l}\Gamma_{2,\,l}$$

we have

$$\Theta_B=\overline{M}_B l\left[\frac{\Gamma_{1,\,l}}{l}-2\frac{\Gamma_{2,\,l}}{l^2}+2\frac{\Gamma_{3,\,l}}{l^3}\right]+\overline{M}_A l\left[\frac{\Gamma_{2,\,l}}{l^2}-2\frac{\Gamma_{3,\,l}}{l^3}\right] \tag{28.10}$$

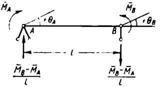
$\bar{M}_A$
θ_A
A
$\bar{M}_B$
θ_B
B
l
$\frac{\bar{M}_B-\bar{M}_A}{l}$
$\frac{\bar{M}_B-\bar{M}_A}{l}$

Fig. 28.3

On the basis of formulas (28.9) and (28.10) we obtain the three moment equation for two adjacent spans of a multispan beam

$$\overline{M}_{n-1}l_n\left[\frac{\Gamma_{2,\,n,\,n}}{l_n^2}-2\frac{\Gamma_{3,\,n,\,n}}{l_n^3}\right]+$$

$$+\overline{M}_n l_n\left[\frac{\Gamma_{1,\,n,\,n}}{l_n^3}-2\frac{\Gamma_{2,\,n,\,n}}{l_n^2}+2\frac{\Gamma_{3,\,n,\,n}}{l_n^3}\right]+$$

$$+\overline{M}_n l_{n+1}\cdot 2\frac{\Gamma_{3,\,n+1,\,n+1}}{l_{n+1}^3}+\overline{M}_{n+1}l_{n+1}\left(\frac{\Gamma_{2,\,n+1,\,n+1}}{l_{n+1}^2}-2\frac{\Gamma_{3,\,n+1,\,n+1}}{l_{n+1}^3}\right)=$$

$$=\Theta^0_{n,\,n+1}-\Theta^0_{n,\,n} \tag{28.11}$$

where

$\Gamma_{1,\,n,\,n}$, $\Gamma_{2,\,n,\,n}$, $\Gamma_{3,\,n,\,n}$ = first, second and third integrals of the flexibility at support n of span n (the first subscript is the order of the integral; the second, the number of the support, the third, the span number), i.e.. the finite values of integrals in span n when $x=l_n$

$\Gamma_{2,\,n+1,\,n+1}$, $\Gamma_{3,\,n+1,\,n+1}$ = second and third integrals of the flexibility evaluated for support $(n+1)$ of span, i.e.. the finite values of integrals in span $(n+1)$ when $x=l_{n+1}$

$\Theta^0_{n,\,n+1} - \Theta^0_{n,\,n}$ = difference between angles determined at support n for spans $(n+1)$ and n caused by span loads applied to corresponding simple beams

The three moment equation may be transformed into another form on the basis of formulas (28.4)-(28.8) by substituting higher moments for flexibility integrals

$$\overline{M}_{n-1} l_n \left(\frac{S_{n,\,n}}{l_n^2} - \frac{I_{n,\,n}}{l_n^3}\right) + \overline{M}_n l_n \left(\frac{F_{n,\,n}}{l_n} - 2\frac{S_{n,\,n}}{l_n^2} + \frac{I_{n,\,n}}{l_n^3}\right) +$$
$$+ \overline{M}_n l_{n+1} \frac{I_{n+1,\,n+1}}{l_{n+1}^3} + \overline{M}_{n+1} l_{n+1} \left(\frac{S_{n+1,\,n+1}}{l_{n+1}^2} - \frac{I_{n+1,\,n+1}}{l_{n+1}^3}\right) = \Theta^0_{n,\,n+1} - \Theta^0_{n,\,n} \tag{28.12}$$

where $\overline{M}_{n-1}$, $\overline{M}_n$, $\overline{M}_{n+1}$ = moments at supports of a multispan beam

$F_{n,\,n}$, $S_{n,\,n}$, $I_{n,\,n}$ = area of the flexibility diagram evaluated over span n, static moment and moment of inertia of the flexibility diagram for span n with respect to the vertical line passing through support n

$S_{n+1,\,n+1}$, $I_{n+1,\,n+1}$ = static moment and moment of inertia of the flexibility diagram for span $(n+1)$ with respect to the vertical line passing through support $(n+1)$

To simplify the practical application of the three moment equation, expressions (28.11) or (28.12) for the most often used loads may be derived for the angles at supports.

Several particular types of loading will be now examined.

1. If a force couple $\overline{M}$ is applied at distances a and $(l - a)$ from the supports (Fig. 27.1), then from the condition that at the right support $v = 0$ and $x = l$ formula (22.6) gives

$$\Theta_A l - 2\left(-\frac{\overline{M}}{l}\right)\Gamma_{3,\,l} - \overline{M}\Gamma_{2,\,a} - \overline{M}\Gamma_{1,\,a}(l-a) = 0$$

hence,

$$\Theta_A = \overline{M} l \left[-2\frac{\Gamma_{3,\,l}}{l^3} + \frac{\Gamma_{2,\,a}}{l^2} + \frac{(l-a)}{l}\cdot\frac{\Gamma_{1,\,a}}{l}\right] \tag{28.13}$$

To be exact, the section in which this formula determines the angle is located at an infinitesimal distance to the left of point B, so that

$$M_B = 0;\quad Q_B = -\frac{\overline{M}}{l};\quad q_B = 0;\quad k_B = 0$$

Since there are no distributed loads on the beam, $q_l = 0$, $k_l = 0$.

The left-hand reaction of the support is not included in (28.13) because it has a zero abscissa and all the corresponding integrals Γ

are zero as the vertical axis passing through the left support does not intercept the flexibility diagram.

Finally, to the left of point B, there is one force couple $\overline{M}$ (at point C with distances a and $l - a$ from the supports), for which the factors are $\Gamma_{2,\ a}$ and $\Gamma_{1,\ a}\,(l - a)$; for point B the abscissa x is equal to l.

Substitution of higher moments for the integrals Γ in (28.13) gives

$$\Theta_A = \overline{M} l \left[-\frac{2}{l^3} \cdot \frac{M_{B^2}}{2} + \frac{M_{C^1}}{l^2} + \frac{l-a}{l^2} F_{A\text{-}C} \right] \qquad (28.14)$$

where M_{B^2} = second moment of the total flexibility diagram with respect to vertical axis passing through point B

$F_{A\text{-}C}$ = area of flexibility diagram between sections A and C

M_{C^1} = static moment of the above mentioned area with respect to vertical axis passing through point C

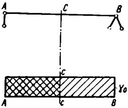

Fig. 28.4

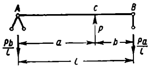

Fig. 28.5

The following expressions are derived from general formulas (28.13) and (28.14) for a uniform beam (Fig. 28.4)

$$\Gamma_{1.\ x} = M_{X^0} = \frac{x}{EI_0}; \quad \Gamma_{2.\ x} = \frac{M_{X^1}}{1} = \frac{x^2}{2EI_0}$$

$$\Gamma_{3.\ x} = \frac{M_{X^2}}{2} = \frac{x^3}{3!\ EI_0}; \quad \Gamma_{4.\ x} = \frac{M_{X^3}}{3!} = \frac{x^4}{4!\ EI_0}$$

$$\Gamma_{5.\ x} = \frac{M_{X^4}}{4} = \frac{x^5}{5!\ EI_0}$$

For $x = l$ and $x = a$, the necessary particular values are given b

$$\Gamma_{3.\ l} = \frac{M_{B^2}}{2} = \frac{l^3}{3!\ EI_0}; \quad \Gamma_{2.\ a} = M_{C^1} = \frac{a^2}{2EI_0}\ \ldots$$

$$\Gamma_{1.\ a} = F_{A\text{-}C} = \frac{a}{EI_0}$$

then

$$\Theta_A = \frac{\bar{M}}{lEI_0}\left(-\frac{l^2}{3} + la - \frac{a^2}{2}\right) \qquad (28.15)$$

This formula may be used in determining the rotation angles of a beam at its left support if the beam is uniform and force couples are applied to its ends; this is performed by assuming $a = 0$ and $a = l$.

The angle at the right support is determined by (22.3) when $x = l$ by substituting (28.13) for Θ_A

$$\begin{aligned}\Theta_B &= \Theta_A + \left[-\frac{\bar{M}}{l}\Gamma_{2,\,l}\right] - [\bar{M}\Gamma_{1,\,a}] = \\ &= \bar{M}\left[-\frac{2}{l^2}\Gamma_{3,\,l} - \frac{1}{l}\Gamma_{2,\,l} + \frac{\Gamma_{2,\,a}}{l} - \frac{a}{l}\Gamma_{1,\,a}\right] = \\ &= \bar{M}\left[-\frac{2}{l^2}\cdot\frac{M_{B^2}}{2} - \frac{1}{l}M_{B^1} + \frac{M_{C^1}}{l} - \frac{a}{l}F_{A\text{-}C}\right] \qquad (28.16)\end{aligned}$$

2. If a point load P is applied to a beam (Fig. 28.5), then by taking $X = B$, i.e., $x = l$ from formula (22.6) we obtain

$$v_B = \Theta_A l + \left[-2\left(P\frac{a}{l}\right)\Gamma_{3,\,l}\right] - [-2P\Gamma_{3,\,a}] - [-P\Gamma_{2,\,a}(l-a)] = 0$$

hence,

$$\begin{aligned}\Theta_A &= \frac{P}{l}\left[2\frac{a}{l}\Gamma_{3,\,l} - 2\Gamma_{3,\,a} - \Gamma_{2,\,a}b\right] = \\ &= \frac{P}{l}\left[2\frac{a}{l}\cdot\frac{M_{B^2}}{2} - 2\frac{M_{C^2}}{2} - bM_{C^1}\right] \qquad (28.17)\end{aligned}$$

Then from formula (22.3) when $x = l$

$$\begin{aligned}\Theta_B &= \Theta_A + \left[-P\frac{a}{l}\Gamma_{2,\,l}\right] - [-P\Gamma_{2,\,a}] = \\ &= P\left[2\frac{a}{l^2}\Gamma_{3,\,l} - \frac{a}{l}\Gamma_{2,\,l} - \frac{2}{l}\Gamma_{3,\,a} + \frac{a}{l}\Gamma_{2,\,a}\right] = \\ &= P\left[2\frac{a}{l^2}\cdot\frac{M_{B^2}}{2} - \frac{a}{l}M_{B^1} - \frac{2}{l}\cdot\frac{M_{C^2}}{2} + \frac{a}{l}M_{C^1}\right] \qquad (28.18)\end{aligned}$$

If the beam is subjected to a system of point loads, the angular displacements Θ_A and Θ_B are determined as a sum of the right-hand parts of (28.17) and (28.18)

3. If a continuously distributed load is applied to the entire beam (Fig. 28.6) then from formulas (22.6) and (22.3) we

obtain

$$v_D=\Theta_A l+\left[-2\left(+\frac{ql}{2}\right)\Gamma_{3,\,l}+3q\Gamma_{4,\,l}\right]=0 \qquad (28.19)$$

$$\Theta_A=q\left[\Gamma_{3,\,l}-\frac{3}{l}\Gamma_{4,\,l}\right]=q\left[\frac{M_{B2}}{2}-\frac{3}{l}\cdot\frac{M_{B3}}{3!}\right]$$

$$\Theta_B=\Theta_A+\left[-\left(\frac{ql}{2}\right)\Gamma_{2,\,l}+q\Gamma_{3,\,l}\right]=q\left[-\frac{l}{2}\Gamma_{2,\,l}+2\Gamma_{3,\,l}-\frac{3}{l}\Gamma_{4,\,l}\right]=$$

$$=q\left[-\frac{l}{2}M_{B1}+2\frac{M_{B2}}{2}-\frac{3}{l}\cdot\frac{M_{B2}}{3!}\right] \qquad (28.20)$$

4. If a continuously distributed load is applied to the part $CB = b$

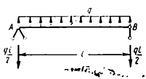

Fig. 28.6

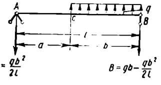

Fig. 28.7

of the beam (Fig. 28.7) then from formulas (22.6) and (22.3) we obtain

$$v_D=\Theta_A l+\left[-2\left(qb-\frac{qb^2}{2l}\right)\Gamma_{3,\,l}+3q\Gamma_{4,\,l}\right]-$$

$$-[3q\Gamma_{4,\,a}]-[q\Gamma_{3,\,a}(l-a)]=0$$

$$\Theta_A=-\frac{q}{l}\left[-2\left(b-\frac{b^2}{2l}\right)\Gamma_{3,\,l}+3\Gamma_{4,\,l}-3\Gamma_{4,\,a}-b\Gamma_{3,\,a}\right] \qquad (28.21)$$

$$\Theta_B=\Theta_A+\left[-\left(qb-\frac{qb^2}{2l}\right)\Gamma_{2,\,l}+q\Gamma_{3,\,l}\right]-$$

$$-[q\Gamma_{3,\,a}]=\frac{q}{l}\left[-\left(b-\frac{b^2}{2l}\right)l\Gamma_{2,\,l}+\right.$$

$$\left.+\left(2b-\frac{b^2}{l}+l\right)\Gamma_{3,\,l}-3\Gamma_{4,\,l}-a\Gamma_{3,\,a}+3\Gamma_{4,\,a}\right] \qquad (28.22)$$

5. If a uniformly increasing load is applied to the part $CB = b$ of the beam (Fig. 28.8) then from formulas (22.6) and (22.3) we obtain

$$\Theta_A=-\frac{1}{l}[2B\Gamma_{3,\,l}+3kb\Gamma_{4,\,l}-4k\Gamma_{5,\,l}+4k\Gamma_{5,\,a}+k\Gamma_{4,\,a}b] \qquad (28.23)$$

$$\Theta_B=\Theta_A+B\Gamma_{2,\,l}+kb\Gamma_{3,\,l}-k\Gamma_{4,\,l}+k\Gamma_{4,\,a} \qquad (28.24)$$

Angles at supports of a beam under the influence of compound loads (Fig. 28.9) may be determined either by summation of solutions obtained for separate particular cases, or by applying formulas (22.6) and (22.3) directly.

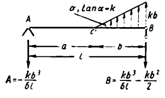

Fig. 28.8

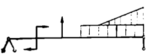

Fig. 28.9.

29. MOHR'S INTEGRALS FOR NON-UNIFORM BEAMS

Formulas derived in Sec. 22 for determining displacements in non-uniform beams may be applied to calculate integrals of the following type

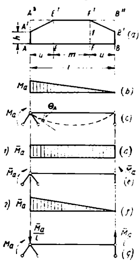

Fig. 29.1

$$\int iM\overline{M}\,dx$$

for non-uniform rods.

For example, consider a beam shown in Fig. 22.4, whose i diagram is presented in Fig. 29.1*a*. Suppose that the diagram of bending moment M_a (Fig. 29.1*b*) must be multiplied by several given diagrams.

The M_a diagram may be a result of loading a simple beam with a force couple at the left support (Fig. 29.1*c*). Such generalized forces correspond to the angle at the left support as a generalized displacement. Therefore to ascertain the correctness of the calculated integral $\int iM\overline{M}dx$ or any new integral for including in pertinent table we assume that the load from which the moment diagram M_a was obtained is restored; then the angle Θ_A on the left-hand support is calculated and multiplied by M_a.

Examine several particular loads.

1. The bending moment diagram is a result of loading of the beam by two equal force couples at supports (Fig. 29.1*d*).

Since the flexibility diagram is a polygon, calculations will be performed by the third method described in Sec. 23.

From the condition that the deflection at the right-hand support is zero according to formula (22.19) we obtain

$$EI_C v_B = EI_C \Theta_A l + M_B M_{B1} = 0 \tag{29.1}$$

in which M_B = bending moment at support B ($M_B = \overline{M}_a$)

M_{B1} = first (static) moment of the total i diagram with respect to vertical axis passing through point B (Fig. 29.1*a*)

Mohr's integral is determined by

$$\int M\overline{M}i\,dx = EI_C \int \frac{M\overline{M}}{EI}\,dx = -EI_C \Theta_A M_a \tag{29.2}$$

The area of the i diagram is calculated as a difference between the area of a rectangle $AA''B''B$ (i.e., $1 \cdot l$) and of two equal triangles, $A'A''E'$ and $F'B''B'$

$$F = 1 \cdot l - \frac{2(1-n)}{2}u = l\left[1-(1-n)\frac{u}{l}\right] \tag{29.3}$$

The static moment of the i diagram is

$$M_{B1} = F\frac{l}{2} = \frac{l^2}{2}\left[1-(1-n)\frac{u}{l}\right] \tag{29.4}$$

Substitution of (29.1) and (29.4) in (29.2) gives

$$I = -EI_C\Theta_A M_a = \frac{M_a M_B M_{B1}}{l} = M_a \overline{M}_a \frac{M_{B1}}{l} =$$

$$= \frac{M_a \overline{M}_a l}{2}\left[1-(1-n)\frac{u}{l}\right] \tag{29.5}$$

2. Here the bending moment diagram is a triangle of height $\overline{M}_a$ (Fig. 29.1*f*) which is a result of loading the left support by force couple (Fig. 29.1*g*).

Since $M_B = 0$ and $Q_B = -\frac{\overline{M}_a}{l}$, it follows from (22.19), that

$$EI_C v_B = EI_C \Theta_A l + 2\frac{\overline{M}_a}{l} \cdot \frac{M_{B2}}{2!} \tag{29.6}$$

Flexibility moments in (29.6) are determined from (22.23)

$$\frac{M_{B2}}{2} = n\frac{l^3}{3!} + \frac{1-n}{u} \cdot \frac{l^4}{4!} - \frac{1-n}{u} \cdot \frac{(l-u)^4}{4!} - \frac{(l-n)}{u} \cdot \frac{u^4}{4!}$$

After collecting similar terms we obtain

$$I = -EI_C\Theta_A M_a = 2\frac{\overline{M}_a}{l^2} M_a \frac{M_{B2}}{2} =$$

$$= \frac{M_a \overline{M}_a l}{6}\left[2+(1-n)\frac{u}{l}\left(-3+2\frac{u}{l}-\frac{u^2}{l^2}\right)\right] \tag{29.7}$$

3. A force couple applied to the right support gives a triangular bending moment diagram with its apex pointing to the left (Fig. 29.2).

In this case, according to (23.19)

$$0 = EI_C\Theta_A l + \overline{M}_a M_{B1} - 2\,\frac{\overline{M}_a}{l}\cdot\frac{M_{B2}}{2}$$

and it follows from (29.2) that

$$1 = -EI_C\Theta_A M_a = \frac{2M_a\overline{M}_a}{l^2}\left(\frac{l}{2}M_{B1} - \frac{M_{B2}}{2}\right) \qquad (29.8)$$

where M_{B1} and $\frac{M_{B2}}{2}$ are determined from (22.23).

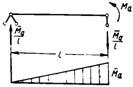

Fig. 29.2

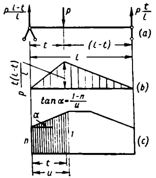

Fig. 29.3

4. A point load P applied at distance t from the left support (Fig. 29.3*a*) gives a bending moment diagram in the form of a triangle (Fig. 29.3*b*) with height $P\frac{t(l-t)}{l}$ at the point of application.

According to (22.19)

$$EI_C\Theta_A l + \left[-2Q_B\frac{M_{B2}}{2!}\right] - \left[-2(-P)\frac{M_{t2}}{2!}\right] -$$
$$-[-(-P)(l-t)M_{t1}] = 0 \qquad (29.9)$$

When deriving this formula from (22.19) it is taken into account that $M_B = 0$, $b_i = t$, $x = l$ and force P, according to its direction, has a minus sign; besides, in (29.9)

$$Q_B = -P\frac{t}{l}$$

The value of $\frac{M_{B2}}{2!}$ may be calculated from formula (22.23). Moments M_{t1} and $\frac{M_{t2}}{2!}$ are evaluated over the shaded part of the i diagram shown in Fig. 29.3*c* with respect to the point of application of the load. The shaded trapezium may be substituted by a rectangle $n \times t$ and a triangle with a slope $k = \tan \alpha = \frac{1-n}{u}$. That is why the area of the trapezium and its moments are expressed by

$$M_{t0} = F_t = nt + \frac{1-n}{u} \cdot \frac{t^2}{2}; \ M_{t1} = n\frac{t^2}{2} + \frac{1-n}{u} \cdot \frac{t^3}{3!}$$

$$\frac{M_{t2}}{2!} = n\frac{t^3}{3!} + \frac{1-n}{u} \cdot \frac{t^4}{4!}$$

Substituting the calculated values into formula (29.9) we obtain the value of $EI_C\Theta_A$, then formula (29.2) will give the unknown integral.

The integrals are calculated similarly when the point load is applied within the other two sections of the i diagram.

5. The bending moment diagram $\overline{M}_a$ for a uniformly distributed load applied to the entire beam is a symmetrical parabola with an ordinate $\overline{M}_a = \frac{ql^2}{8}$ at mid-span (Fig. 29.4).

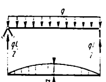

Fig. 29.4

For this case formula (22.19) has the form

$$0 = EI_C\Theta_A l + \left[-2Q_B\frac{M_{B2}}{2} + 3q_B\frac{M_{B3}}{3!}\right] \qquad (29.10)$$

where $Q_B = -\frac{ql}{2}$, $q_B = -q$, and higher moments are determined by (22.23). The integral is calculated by solving (29.10) for $EI_C\Theta_A$ and substituting the result into (29.2).

When the initial M diagram is outlined by any figure other than a triangle, the integral $\int iM\overline{M}dx$ may be determined likewise.

Chapter VI
Beams on Elastic Foundation

30. GENERAL

Initially all theories on designing beams were based on Fuss-Vinkler's (or Shvedler-Zimmerman) hypothesis which assumed the depression of a foundation to be proportional to the elastic reaction of the foundation at any point of a beam. This means that deflections (depressions) take place only within the length of the beam. The authors using these designing methods were concerned mainly with developing simple practical methods of calculations, in compiling reference tables and nomographs, in composing diagrams and in developing other methods to facilitate the calculation technique.

Methods of designing beams on the basis of Vinkler's hypothesis were developed in detail by a number of scientists, however, these methods had considerable disadvantages.

A number of published scientific works amended the main precondition made by Vinkler; in these works the depressions were allowed to spread beyond the limits of beams lying on continuous elastic foundations.

For example, Vighart introduced a certain decreasing exponential function for this purpose. The choice was rather arbitrary, although it made it possible to solve a uniform beam having a uniform load distributed along the entire length. According to Vinkler, such a beam should not bend. However, a certain arbitrariness of Vighart's theory and the considerable difficulties in solving engineering problems led to other ways of developing a general theory for the evaluation of beams on elastic foundations.

Investigators began to realize that the theory of elasticity may be applied not only to the beam, but also to the elastic foundation. New methods based on the theory of elasticity (theory of elastic half-plane and elastic half-space) still had deficiencies, so the vinklerian beam design is still used in practice. This happens because Vinkler's hypothesis is used to some extent for calculating not only beams, but also shells and tanks.

Therefore, the following considerations for the maximum possible simplification of the calculation technique are based on elementary dependencies between depression and pressure on elastic foundation. The solutions are obtained by the moment-operational method for uniform and non-uniform beams on foundations with constant and varying elasticity subjected to any static and kinematic load. These solutions allow for possible breaks and steps in the flexural curve, both point and distributed.

Problems examined below were chosen to demonstrate the practical applications of various methods rather than to solve any particular problem in beam designing.

In order to show the advantages of applying the higher moment theory, problems solved before by non-momental methods will be examined herein.

31. PRISMATIC BEAMS ON FOUNDATION OF CONSTANT RIGIDITY

If the depression of a beam is assumed to be proportional to the pressure on the foundation, a differential equation of the fourth order can be derived

$$EI\frac{d^4v}{dx^4} + kv = q \tag{31.1}$$

Let $k = EI\alpha^2$ and $q = \frac{d^2M}{dx^2} = M_{-1}$, then

$$EI\frac{d^4v}{dx^4} + \alpha^2 EIv = M_{-1} \tag{31.2}$$

This equation can be solved by the moment-operational method after several iterations.

In the first approximation, the elastic foundation is disregarded by assuming $\alpha^2 EIv = 0$; therefore

$$EI\frac{d^4v}{dx^4} = M_{-1}$$

hence,

$$EIv = \frac{M_3}{3!}$$

This solution is substituted into equation (31.2), which may be written as

$$EI\frac{d^4v}{dx^4} = M_{-1} - \alpha^2 EIv \tag{31.3}$$

or

$$EI\frac{d^4v}{dx^4} = M_{-1} - \alpha^2\frac{M^3}{3!}$$

The fourth integral gives the next approximation

$$EIv = \frac{M_3}{3!} - \alpha^2\frac{M_7}{7!}$$

Successive approximations give an infinite series of moments

$$EIv = \frac{M_3}{3!} - \alpha^2\frac{M_7}{7!} + \alpha^4\frac{M_{11}}{11!} - \alpha^6\frac{M_{15}}{15!} + \dots \qquad (31.4)$$

Convergence of this series may be proved by expanding it for a point load P applied to a section at distance u from the point where deflection is to be determined; then

$$EIv = P\left[\frac{u^3}{3!} - \alpha^2\frac{u^7}{7!} + \alpha^4\frac{u^{11}}{11!} - \alpha^6\frac{u^{15}}{15!} + \dots\right] \qquad (31.5)$$

General expressions for the n-th and $(n+1)$-th terms are given by

$$|u_n| = \alpha^{2n-2}\frac{u^{4n-1}}{(4n-1)!}\,;\quad u_{n+1} = \alpha^{2n}\frac{u^{4n+3}}{(4n+3)!}$$

hence, their ratio is

$$\frac{|u_{n+1}|}{|u_n|} = \frac{\alpha^2u^4}{4n(4n+1)(4n+2)(4n+3)} \qquad (31.6)$$

and its limit is

$$\lim_{n\to\infty}\frac{|u_{n+1}|}{u_n} = 0$$

Consequently, formula (31.5) is a converging monotonously decreasing alternating series.

Formulas (31.4) and (31.5) give the exact solution when the number of terms is infinite; the first terms of both formulas correspond to a beam without an elastic foundation (with rigid supports only). Therefore, by limiting these formulas to several terms, intermediate solutions are obtained for (31.1), i.e., solutions which fall between the exact solution of a beam on a vinklerian foundation and an exact solution of a beam without an elastic foundation.

However, series (31.4) does not always converge sufficiently to limit it to two terms, as for the case of symmetrically loaded symmetrical beams, when the reference point may be located at mid-span; in other cases three terms are required in the formula. K. Khayashi examined the first problem and compared the exact solution of a beam on elastic foundation under the influence of a point load at mid-span and the approximate solution of the problem given by two terms in a series; the bending moments in these two solutions were

30.15P and 30.05P, respectively, which means that they do not differ significantly.

Limiting the series to three terms, from (31.4) we obtain the following reference formulas

$$EIv = \frac{M_3}{3!} - \alpha^2 \frac{M_7}{7!} + \alpha^4 \frac{M_{11}}{11!} \tag{31.7}$$

$$EI\Theta = \frac{M_2}{2} - \alpha^2 \frac{M_6}{6!} + \alpha^4 \frac{M_{10}}{10!} \tag{31.8}$$

$$M = M_1 - \alpha^2 \frac{M_5}{5!} + \alpha^4 \frac{M_9}{9!} \tag{31.9}$$

$$Q = M_0 - \alpha^2 \frac{M_4}{4!} + \alpha^4 \frac{M_8}{8!} \tag{31.10}$$

These formulas may be applied to solve particular problems by taking into account initial parameters.

As mentioned above, the vinklerian beam is of considerable practical interest. However, since all the problems associated with this design are studied in detail, formulas (31.7)-(31.10) will not be applied to any particular problem. We shall only show a way of eliminating difficulties associated with solving beams subjected to discontinuous loads by means of higher moments. This may be useful in solving other problems connected with the approximate solutions of different equations for beams under compound loadings.

K. Khayashi solves the following differential equation

$$\frac{d^4y}{d\xi^4} + 4y = 0 \tag{31.11}$$

by a general expression

$$y = f(\xi) = A_1X_1 + A_2X_2 + A_3X_3 + A_4X_4 \tag{31.12}$$

where

$$X_1 = 1 - \frac{\xi^4}{4!}4 + \frac{\xi^8}{8!}4^2 - \frac{\xi^{12}}{12!}4^3 + \dots \tag{31.13}$$

$$X_2 = \xi - \frac{\xi^5}{5!}4 + \frac{\xi^9}{9!}4^2 - \frac{\xi^{13}}{13!}4^3 + \dots \tag{31.14}$$

$$X_3 = \frac{\xi^2}{2} - \frac{\xi^6}{6!}4 + \frac{\xi^{10}}{10!}4^2 - \frac{\xi^{14}}{14!}4^3 + \dots \tag{31.15}$$

$$X_4 = \frac{\xi^3}{3!} - \frac{\xi^7}{7!}4 + \frac{\xi^{11}}{11!}4^2 - \frac{\xi^{15}}{15!}4^3 + \dots \tag{31.16}$$

and the constants of integration are given by

$$A_1 = f(0) = EIv_0; \quad A_2 = f'(0) = EI\Theta_0$$

$$A_3 = f''(0) = M_0; \quad A_4 = f'''(0) = Q_0$$

Equation (31.2) for a point load is

$$\frac{d^4v}{dx^4}+\alpha^2v=0 \tag{31.17}$$

Comparing formulas (31.11) and (31.17) we see that the solution of (31.17) can be obtained from equation (31.11) by substituting α^2 for 4, therefore

$$X_4=\frac{x^3}{3!}-\alpha^2\frac{x^7}{7!}+\alpha^4\frac{x^{11}}{11!}-\dots \tag{31.18}$$

Since X_4 refers to the influence of the initial point load, and the orders of moments correspond to the powers of arms of point loads when moment symbolics are applied, it follows from (31.18) that

$$EIv=\frac{M_3}{3!}-\alpha^2\frac{M_7}{7!}+\alpha^4\frac{M_{11}}{11!}-\dots \tag{31.19}$$

which is the previously obtained formula (31.4). Therefore, it may be used for solving beams under various loadings, including discontinuous ones.

We shall apply the above described method of approximative integration of differential equations in terms of infinite moment series to uniform and non-uniform beams on elastic foundations for the purpose of obtaining new general solutions.

32. PRISMATIC BEAMS ON FOUNDATION OF LINEAR RIGIDITY

For a prismatic beam of rigidity EI, let the elasticity coefficient of the foundation change according to a linear law, so that at any section X with abscissa x (Fig. 32.1) this coefficient is

$$k_X=k_0+k_1x \tag{32.1}$$

The differential equation for the flexural curve is given by

$$EI\frac{d^4v}{dx^4}=q-k_Xv \tag{32.2}$$

or

$$EI\frac{d^4v}{dx^4}+\frac{k_0+k_1x}{EI}\cdot EIv=M_{-1} \tag{32.3}$$

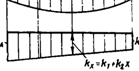

Fig. 32.1

Denoting

$$k_0=\alpha EI;\quad k_1=\beta EI \tag{32.4}$$

we obtain

$$EI\frac{d^4v}{dx^4}=M_{-1}-\alpha EIv-\beta xEIv \tag{32.5}$$

Assuming for the first approximation that $\alpha = \beta = 0$ (i.e., a beam without an elastic foundation) the following is derived from (32.5)

$$EIv = \frac{M_3}{3!} \tag{32.6}$$

as for a simply supported beam.

By allowing for the influence of the elastic foundation, more exact solutions are found by the method of successive approximation from the equation

$$EI\frac{d^4v}{dx^4} = -\alpha EIv - \beta xEIv \tag{32.7}$$

This equation is integrated four times by means of formula (18.13)

$$\int\int\int\int \frac{x^m}{m!}\cdot\frac{M_n}{n!}\,dx^4 =$$

$$= \sum_{h=0}^{h=m} (-1)^h \frac{(k+1)(k+2)(k+3)}{1\cdot 2\cdot 3} \frac{x^{m-k}}{(m-k)!}\cdot\frac{M_{n+k+4}}{(n+k+4)!} \tag{32.8}$$

Formulas and methods outlined in Sec. 19 make it possible to arrive at an expression for EIv by successive approximations.

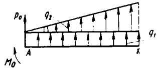

Fig. 32.2

In a particular case, when only the initial values v_0, Θ_0, $\overline{M}_0$, $Q_0 = P_0$ are considered as well as continuous loads (both uniform and uniformly increasing) applied to the entire beam (Fig. 32.2), the fifth approximation for EIv is obtained by summing the right-hand parts of the formulas discussed earlier. The following expression for the n-th moment, divided by $n!$, must be substituted in the resulting formula and all the given loads must be taken into account, including the kinematic factor (v_0 and Θ_0)

$$\frac{M_n}{n!} = EIv_0\frac{x^{n-3}}{(n-3)!} + EI\Theta_0\frac{x^{n-2}}{(n-2)!} +$$

$$+ \overline{M}_0\frac{x^{n-1}}{(n-1)!} + P_0\frac{x^n}{n!} + q_1\frac{x^{n+1}}{(n+1)!} + q_2\frac{x^{n+2}}{(n+2)!} \tag{32.9}$$

In order to obtain a complete solution for any arbitrary loads, point or distributed, supplementary terms must be introduced in the moment expression for every order.

This method may be applied to other problems of designing beams on elastic foundations.

The general procedure of solving differential equations by successive approximations is similar to the method of solving general differential equations and involves the following:

(1) All terms except the one containing the higher derivative are assumed equal to zero; the first approximation is derived by solving the resulting simple equation in terms of moment functions.

(2) All terms except the one containing the higher derivative are transferred to the right-hand side of the equation and replaced by expressions derived in the previous step; this new equation is then integrated to give the next approximation for the function and its derivatives, etc.

The recommended method of successive approximations gives an infinite moment series and is not the only way of solving differential equations.

Another possible method consists in solving differential equations in terms of moment series with indeterminate coefficients. The values of the coefficients are obtained by solving a system of linear algebraic equations, which is formed by substituting the assumed function and its derivatives in the differential equation.

For practical convenience of calculating the coefficients it is necessary to study formula

$$y = \sum_{m=0,\,1,\,2\,\dots} \; \sum_{n=0,\,1,\,2\,\dots} a_{m,\,n} \frac{x^m}{m!} \cdot \frac{M_n}{n!} \qquad (32.10)$$

for the purpose of setting many coefficients equal to zero.

In some engineering problems, such as evaluating beams on elastic foundation according to the elasticity theory, the solution obtained for a point load (the influence line) must be expanded either to infinite power series (to derive formulas with a small number of terms) or to finite power series, which approximate the complex analytical expression. This will lead to quick and easy solutions in terms of higher moments for any load.

33. BEAMS ON FOUNDATION OF HYPERBOLIC RIGIDITY

Examine a beam on elastic foundation; suppose the rigidity of the beam follows a hyperbolic law (Fig. 33.1).

Let

$$EI_x = \frac{EI_0}{x} \qquad (33.1)$$

where EI_0 is the rigidity of a section at distance $x = 1$ from the left end, which is taken as an origin of coordinates.

Denoting as before

$$k = \alpha^2 EI_0 \qquad (33.2)$$

we write the differential equation of the flexural curve of a non-uniform beam

$$\frac{d^2}{dx^2}\left(EI_x\frac{d^2v}{dx^2}\right)+kv=\frac{d^2M}{dx^2} \tag{33.3}$$

or, for the considered case

$$EI_0\frac{d^2}{dx^2}\left(\frac{1}{x}\cdot\frac{d^2v}{dx^2}\right)+\alpha^2EI_0v=\frac{d^2M}{dx^2} \tag{33.4}$$

The first approximation is given by the following differential equation, which is derived from (33.4) by disregarding the second term in the left-hand part

$$EI_0\frac{d^2}{dx^2}\left(\frac{1}{x}\cdot\frac{d^2v}{dx^2}\right)=\frac{d^2M}{dx^2}$$

The second integral of this equation is given by

$$EI_0\frac{1}{x}\cdot\frac{d^2v}{dx^2}=M_1$$

hence,

$$EI_0\frac{d^2v}{dx^2}=xM_1$$

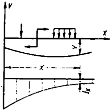

Fig. 33.1

Further successive approximations performed by formulas of Sec. 19 result in

$$EI_0v=\left(x\frac{M_3}{3!}-2\frac{M_4}{4!}\right)-\alpha^2\left(x^2\frac{M_7}{7!}-8x\frac{M_8}{8!}+14\frac{M_9}{9!}\right)+$$
$$+\alpha^4\left(x^3\frac{M_{11}}{11!}-18x^2\frac{M_{12}}{12!}+102x\frac{M_{13}}{13!}-168\frac{M_{14}}{14!}\right) \tag{33.5}$$

This formula may be applied to point loads and force couples, because the beam is non-uniform; additional analysis is required to expand (33.5) completely, so that it takes into account point breaks and steps of the flexural curve (for example, initial angle and deflection).

Suppose the beam carries only a kinematic load and there are no static loads on the beam, i.e., $\frac{d^2M}{dx^2}=0$. Then formula (33.4) transforms into

$$EI_0\frac{d^2}{dx^2}\left(\frac{1}{x}\cdot\frac{d^2v}{dx^2}\right)+\alpha^2EI_0v=0 \tag{33.6}$$

or

$$\frac{d^2}{dx^2}\left(\frac{1}{x}\cdot\frac{d^2v}{dx^2}\right)=-\alpha^2v=-\alpha^2M_0(v)$$

where $M_0(v)=\sum v$ is the moment of zero order of all deflections (point and distributed) to the left of the examined section, i.e., an algebraic sum of applied deflections, which is equal to the actual deflection of the beam at this particular point. Integration of (33.6) gives

$$\frac{1}{x}\cdot\frac{d^2v}{dx^2}=-\alpha^2\frac{M_2(v)}{2!}$$

or

$$\frac{d^2v}{dx^2}=-\alpha^2 x\frac{M_2(v)}{2!}$$

Again integrating twice

$$v=-\alpha^2\left[x\frac{M_4(v)}{4!}-2\frac{M_5(v)}{5!}\right]$$

Other terms of the series are obtained by successive approximations carried out by substitution of this expression into the right-hand part of equation (33.6), then

$$\frac{d^2}{dx^2}\left(\frac{1}{x}\cdot\frac{d^2v}{dx^2}\right)=\alpha^4\left[x\frac{M_4(v)}{4!}-2\frac{M_5(v)}{5!}\right]$$

Repeated integration gives

$$\frac{1}{x}\cdot\frac{d^2v}{dx^2}=\alpha^4\left(x\frac{M_6}{6!}-4\frac{M_7}{7!}\right)(v)$$

$$\frac{d^2v}{dx^2}=\alpha^4\left(x^2\frac{M_6}{6!}-4x\frac{M_7}{7!}\right)(v)$$

$$v=\alpha^4\left(x^2\frac{M_8}{8!}-8x\frac{M_9}{9!}+14\frac{M_{10}}{10!}\right)(v)$$

Hence,

$$v=M_0(v)-\alpha^2\left[x\frac{M_4(v)}{4!}-2\frac{M_5(v)}{5!}\right]+$$
$$+\alpha^4\left[x^2\frac{M_8(v)}{8!}-8x\frac{M_9(v)}{9!}+14\frac{M_{10}(v)}{10!}\right]+\ldots \qquad (33.7)$$

Comparing this formula with (33.5), we see that the next term in the right-hand part of (33.7) is

$$-\alpha^6\left[x^3\frac{M_{12}(v)}{12!}-18x^2\frac{M_{13}(v)}{13!}+102x\frac{M_{14}(v)}{14!}-168\frac{M_{15}(v)}{15!}\right] \qquad (33.7a)$$

The kinematic load (deviations) results in an integral expression with respect to (33.7), so that

$$v=M_1(\Theta)-\alpha^2\left[x\frac{M_5(\Theta)}{5!}-2\frac{M_6(\Theta)}{6!}\right]+$$
$$+\alpha^4\left[x^2\frac{M_9(\Theta)}{9!}-8x\frac{M_{10}(\Theta)}{10!}+14\frac{M_{11}(\Theta)}{11!}\right]+\ldots \qquad (33.8)$$

The complete deflection formula is obtained by summing the right-hand parts of (33.5), (33.7) and (33.8); this formula will allow for all types of loads.

For example, an expression for the deflection of a beam of section X shown in Fig. 33.2 is given by

$$
\begin{aligned}
EI_0v = EI_0v_0\left(1-\alpha^2\frac{3x^5}{5!}+\alpha^4\frac{24x^{10}}{10!}-\dots\right)+\\
+EI_0\Theta_0\left(x-\alpha^2\frac{4x^6}{6!}+\alpha^4\frac{36x^{11}}{11!}-\dots\right)+\\
+M_0\left(\frac{x^3}{3!}-\alpha^2\frac{6x^8}{8!}+\alpha^4\frac{66x^{13}}{13!}-\dots\right)+\\
+P_0\left(2\frac{x^4}{4!}-\alpha^2\frac{14x^9}{9!}+\alpha^4\frac{168x^{14}}{14!}-\dots\right)+\\
+q_1\left(3\frac{x^5}{5!}-\alpha^2\frac{24x^{10}}{10!}+\alpha^4\frac{312x^{15}}{15!}-\dots\right)+\\
+q_2\left(4\frac{x^6}{6!}-\alpha^2\frac{36x^{11}}{11!}+\alpha^4\frac{504x^{16}}{16!}-\dots\right)
\end{aligned}
\qquad (33.9)
$$

If the flexibility of a beam is expressed by an algebraic polynomial with a few terms, some other designing method may be more easily applied than the one outlined herein*. To illustrate this, consider the same beam. From the general equation of the flexural curve

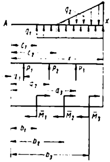

Fig. 33.2

$$\frac{d^2M}{dx^2}=q-kv$$

or

$$\frac{d^2M}{dx^2}=M_{-1}-kv$$

the second derivative is given by

$$M^{\mathrm{IV}}=M_{-3}-k\frac{d^2v}{dx^2} \qquad (33.10)$$

or

$$M^{\mathrm{IV}}=M_{-3}-\frac{k}{EI}M \qquad (33.10a)$$

For this particular case $\frac{1}{EI}=\frac{x}{EI_0}$; denoting $\frac{k}{EI_0}=\alpha^2$ we get

$$M^{\mathrm{IV}}+\alpha^2xM=M_{-3} \qquad (33.11)$$

The first approximation is obtained by disregarding the second term in the left-hand part of this formula

$$M^{\mathrm{IV}}=M_{-3} \text{ and } M=M_1 \qquad (33.12)$$

---

[*] Various methods of solving non-uniform beams on elastic foundation are outlined in the book by A. Krylov.

and successive approximations are determined by

$$M^{\mathrm{IV}} = -\alpha^2 x M \tag{33.13}$$

For the second approximation

$$M^{\mathrm{IV}} = -\alpha^2 x M_1 \tag{33.14}$$

and the fourth integral gives

$$M = -\alpha^2 \left(x \frac{M_5}{5!} - 4 \frac{M_6}{6!} \right) \tag{33.15}$$

Similarly, for the third approximation

$$M^{\mathrm{IV}} = \alpha^4 \left(\frac{x^2}{2} \cdot \frac{M_5}{5!} - 4x \frac{M_6}{6!} \right)$$

$$M = \alpha^4 \left(x^2 \frac{M_9}{9!} - 12x \frac{M_{10}}{10!} + 36 \frac{M_{11}}{11!} \right) \tag{33.16}$$

Thus, after three approximations

$$M = M_1 - \alpha^2 \left(x \frac{M_5}{5!} - 4 \frac{M_6}{6!} \right) + \alpha^4 \left(x^2 \frac{M_9}{9!} - 12x \frac{M_{10}}{10!} + 36 \frac{M_{11}}{11!} \right) \tag{33.17}$$

This bending moment expression may be applied to derive formulas for angles and deflections of a beam under static loads only. The curvature of the beam is given by

$$\frac{1}{\rho} = \frac{M}{EI} = \frac{Mx}{EI_0}$$

hence,

$$\begin{aligned} EI_0 \frac{1}{\rho} = xM = xM_1 - \alpha^2 \left(2 \frac{x^2}{2} \frac{M_5}{5!} - 4x \frac{M_6}{6!} \right) + \\ + \alpha^4 \left(6 \frac{x^3}{3!} \cdot \frac{M_9}{9!} - 24 \frac{x^2}{2} \cdot \frac{M_{10}}{10!} + 36x \frac{M_{11}}{11!} \right) \end{aligned} \tag{33.18}$$

The first and second integrals of (33.18) give the following expressions

$$\begin{aligned} EI_0 \Theta = \left(x \frac{M_2}{2!} - \frac{M_3}{3!} \right) - \alpha^2 \left(2 \frac{x^2}{2} \cdot \frac{M_6}{6!} - 6x \frac{M_7}{7!} + 6 \frac{M_8}{8!} \right) + \\ + \alpha^4 \left(6 \frac{x^3}{3!} \cdot \frac{M_{10}}{10!} - 30 \frac{x^2}{2} \cdot \frac{M_{11}}{11!} + 66x \frac{M_{12}}{12!} - 66 \frac{M_{13}}{13!} \right) \end{aligned} \tag{33.19}$$

and

$$\begin{aligned} EI_0 v = \left(x \frac{M_3}{3!} - 2 \frac{M_4}{4!} \right) - \alpha^2 \left(2 \frac{x^2}{2} \cdot \frac{M_7}{7!} - 8x \frac{M_8}{8!} + \right. \\ \left. + 14 \frac{M_9}{9!} \right) + \alpha^4 \left(6 \frac{x^3}{3!} \cdot \frac{M_{11}}{11!} - 36 \frac{x^2}{2} \cdot \frac{M_{12}}{12!} + \right. \\ \left. + 102x \frac{M_{13}}{13!} - 168 \frac{M_{14}}{14!} \right) \end{aligned} \tag{33.20}$$

In order to take the kinematic loads into account, formula (33.10) is considered, and the initial approximation is obtained by disregarding M_{-3}.

Denoting $\frac{k}{EI_0} = \alpha^2$ we obtain

$$M^{\mathrm{IV}} = -\alpha^2 EI_0 \frac{d^2 v}{dx^2}$$

and the second integral of this equation gives[*]

$$M'' = -\alpha^2 EI_0 v = -\alpha^2 EI_0 M_0(v)$$

$$M = -\alpha^2 EI_0 \frac{M_2(v)}{2} \tag{33.21}$$

Successive approximations are determined by

$$M^{\mathrm{IV}} = +\alpha^4 EI_0 x \frac{M_2(v)}{2} \tag{33.22}$$

the fourth integral of which is given by

$$M = \alpha^4 EI_0 \left(x \frac{M_6}{6!} - 4 \frac{M_7}{7!} \right)(v) \tag{33.23}$$

Repeating the calculations, we obtain

$$M^{\mathrm{IV}} = -\alpha^6 EI_0 \left(2 \frac{x^2}{2} \cdot \frac{M_6}{6!} - 4x \frac{M_7}{7!} \right)(v) \tag{33.24}$$

$$M = -\alpha^6 EI_0 \left(2 \frac{x^2}{2} \cdot \frac{M_{10}}{10!} - 12x \frac{M_{11}}{11!} + 36 \frac{M_{12}}{12!} \right)(v) \tag{33.25}$$

The final solution is given by

$$\frac{M}{EI_0} = \frac{1}{x} \cdot \frac{d^2 v}{dx^2} - \alpha^2 \frac{M_2(v)}{2} + \alpha^4 \left(x \frac{M_6}{6!} - 4 \frac{M_7}{7!} \right)(v) - \\ - \alpha^6 \left(2 \frac{x^2}{2} \cdot \frac{M_{10}}{10!} - 12x \frac{M_{11}}{11!} + 36 \frac{M_{12}}{12!} \right)(v) + \ldots \tag{33.26}$$

Kinematic angular loads are taken into account by raising the orders of moments in formula (33.26) by one unit

$$\frac{M}{EI_0} = \frac{1}{x} \cdot \frac{d\Theta}{dx} - \alpha^2 \frac{M_3(\Theta)}{3!} + \alpha^4 \left(x \frac{M_7}{7!} - 4 \frac{M_8}{8!} \right)(\Theta) - \\ - \alpha^6 \left(2 \frac{x^2}{2!} \cdot \frac{M_{11}}{11!} - 12x \frac{M_{12}}{12!} + 36 \frac{M_{13}}{13!} \right)(\Theta) + \ldots \tag{33.27}$$

---

[*] In (33.21) and in the succeeding formulas $\frac{M_n(v)}{n!}$ designates the n-th moment (divided by $n!$) of a linear kinematic load; when only the initial deflection is considered, this moment is given by $v_0 \frac{x^n}{n!}$.

First derivatives of (33.17), (33.18) and (33.19) give expressions for transverse forces in a beam under static and kinematic loads

$$Q = M_0 - \alpha^2 \left(x \frac{M_4}{4!} - 3 \frac{M_5}{5!} \right) + \\ + \alpha^4 \left(2 \frac{x^2}{2} \cdot \frac{M_8}{8!} - 10x \frac{M_9}{9!} + 24 \frac{M_{10}}{10!} \right) \ldots \qquad (33.28)$$

$$\frac{Q}{EI_0} = -\alpha^2 M_1 (v) + \alpha^4 \left(x \frac{M_5}{5!} - 3 \frac{M_6}{6!} \right) (v) - \\ - \alpha^6 \left(2 \frac{x^2}{2} \cdot \frac{M_9}{9!} - 10x \frac{M_{10}}{10!} + 24 \frac{M_{11}}{11!} \right) (v) \qquad (33.29)$$

$$\frac{Q}{EI} = -\alpha^2 \frac{M_2}{2} (\Theta) + \alpha^4 \left(x \frac{M_6}{6!} - 3 \frac{M_7}{7!} \right) (\Theta) - \\ - \alpha^6 \left(2 \frac{x^2}{2} \cdot \frac{M_8}{8!} - 10x \frac{M_9}{9!} + 24 \frac{M_{10}}{10!} \right) (\Theta) + \ldots \qquad (33.30)$$

If the initial quantities (angle and deflection) are the only kinematic loads applied to beams, then final formulas for bending moments are composed by adding expressions given by (33.26) and (33.27) to (33.17) with the appropriate substitutions

$$\frac{M_n}{n!} (v) = v_0 \frac{(x - x_0)^n}{n!}; \quad \frac{M_n}{n!} (\Theta) = \Theta_0 \frac{(x - x_0)^n}{n!} \qquad (33.31)$$

Formulas (33.31) are composed for initial quantities (deflection v_0 and angle Θ_0) applied to section X_0 with abscissa x_0. When x_0 coincides with the origin of coordinates,

$$\frac{M_n}{n!} (v) = v_0 \frac{x^n}{n!}; \quad \frac{M_n}{n!} (\Theta) = \Theta_0 \frac{x^n}{n!} \qquad (33.32)$$

Formulas (33.31) and (33.32) may be applied to calculate various moments for substitution into (33.26) and (33.27) by assuming various values of n.

Transverse forces are determined similarly: the expressions obtained from the substitution of (33.31) and (33.32) into (33.29) and (33.30) are added to the right-hand part of (33.28).

The final formula for deflections of a beam may be obtained by substituting the expressions from (33.31) and (33.32) into (33.7), (33.7*a*) and (33.8) and adding the resulting expressions to the right-hand part of (33.20).

First derivatives of (33.7), (33.7*a*) and (33.8) are the necessary expressions for the angle at the arbitrary section X with abscissa x

$$\Theta_X = -\alpha^2 \left[x \frac{M_3 (v)}{3!} - \frac{M_4}{4!} (v) \right] + \alpha^4 \left[x^2 \frac{M_7}{7!} (v) - \\ - 6x \frac{M_8}{8!} (v) + 6 \frac{M_9}{9!} (v) \right] + \ldots \qquad (33.33)$$

$$\Theta_X = M_0 (\Theta) - \alpha^2 \left[x \frac{M_4}{4!} (\Theta) - \frac{M_5}{5!} (\Theta) \right] + \\ + \alpha^4 \left[x^2 \frac{M_8}{8!} (\Theta) - 6x \frac{M_9}{9!} (\Theta) + 6 \frac{M_{10}}{10!} (\Theta) \right] + \ldots \qquad (33.34)$$

Substitution of the moment values from (33.31) and (33.32) into (33.33) and (33.34) gives expressions, which may be added to the right-hand part of (33.19) to obtain a complete expression for the angle at an arbitrary section.

34. BEAMS ON ELASTIC FOUNDATION WITH MOMENT REACTION

In the previous sections it was assumed that the elastic foundation had vertical reactions only. In some cases, however, moment reactions of the foundation have to be taken into account as well. In this case the differential equation of the flexural curve for a uniform beam is given by

$$EIv^{\mathrm{IV}} = \gamma EIv'' - \beta EIv + M_{-1} \tag{34.1}$$

where

$$\beta = \frac{k}{EI}; \qquad \gamma = \frac{t}{EI} \tag{34.2}$$

The first approximation is obtained by neglecting the first two terms in the right-hand part of (34.1), as always when solving differential equations of moment functions, i.e., we use only M_{-1}. Then

$$EIv^{\mathrm{IV}} = M_{-1} \quad \text{and} \quad EIv = \frac{M_3}{3!} \tag{34.3}$$

Hence, $EIv'' = M_1$, and from the formula

$$EIv^{\mathrm{IV}} = \gamma EIv'' - \beta EIv \tag{34.4}$$

after substitutions we obtain

$$EIv^{\mathrm{IV}} = \gamma M_1 - \beta \frac{M_3}{3!} \tag{34.5}$$

The second and fourth integrals of this equation are given by

$$\begin{aligned} EIv'' &= \gamma \frac{M_3}{3!} - \beta \frac{M_5}{5!} \\ EIv &= \gamma \frac{M_5}{5!} - \beta \frac{M_7}{7!} \end{aligned} \tag{34.6}$$

Then from formula (34.4) we obtain

$$\begin{aligned} EIv^{\mathrm{IV}} &= \gamma\left(\gamma \frac{M_3}{3!} - \beta \frac{M_5}{5!}\right) - \beta\left(\gamma \frac{M_5}{5!} - \beta \frac{M_7}{7!}\right) = \\ &= \gamma^2 \frac{M_3}{3!} - 2\beta\gamma \frac{M_5}{5!} + \beta^2 \frac{M_7}{7!} \end{aligned}$$

and after integration

$$\left.\begin{aligned} EIv'' &= \gamma^2 \frac{M_5}{5!} - 2\beta\gamma \frac{M_7}{7!} + \beta^2 \frac{M_9}{9!} \\ EIv &= \gamma^2 \frac{M_7}{7!} - 2\beta\gamma \frac{M_9}{9!} + \beta^2 \frac{M_{11}}{11!} \end{aligned}\right\} \tag{34.7}$$

Repeating this procedure to derive successive approximations, we obtain

$$\left.\begin{aligned} EIv^{\mathrm{IV}} &= \gamma^3 \frac{M_5}{5!} - 3\beta\gamma^2 \frac{M_7}{7!} + 3\beta^2\gamma \frac{M_9}{9!} - \beta^3 \frac{M_{11}}{11!} \\ EIv &= \gamma^3 \frac{M_9}{9!} - 3\beta\gamma^2 \frac{M_{11}}{11!} + 3\beta^2\gamma \frac{M_{13}}{13!} - \beta^3 \frac{M_{15}}{15!} \end{aligned}\right\} \qquad (34.8)$$

Further approximations are obtained in the same way. Adding the solution to the ones derived previously, we arrive at the complete formula for deflections

$$\begin{aligned} EIv = {} & \frac{M_3}{3!} + \left(\gamma \frac{M_5}{5!} - \beta \frac{M_7}{7!}\right) + \left(\gamma^2 \frac{M_7}{7!} - \right. \\ & \left. - 2\beta\gamma \frac{M_9}{9!} + \beta^2 \frac{M_{11}}{11!}\right) + \left(\gamma^3 \frac{M_9}{9!} - 3\beta\gamma^2 \frac{M_{11}}{11!} + \right. \\ & \left. + 3\beta^2\gamma \frac{M_{13}}{13!} - \beta^3 \frac{M_{15}}{15!}\right) + \left(\gamma^4 \frac{M_{11}}{11!} - 4\beta\gamma^3 \frac{M_{13}}{13!} + \right. \\ & \left. + 6\beta^2\gamma^2 \frac{M_{15}}{15!} - 4\beta^3\gamma \frac{M_{17}}{17!} + \beta^4 \frac{M_{19}}{19!}\right) + \dots \end{aligned} \qquad (34.9)$$

Chapter VII

Beams Under Combined Flexure and Compression

35. GENERAL

This chapter presents applications of the moment-operational method to strength, rigidity and stability evaluation of beams subjected to combined compression and flexure, and further development of the method is outlined for more complex problems.

The computation of beams subjected to longitudinal and transverse loads is, in the present day, of great practical importance; that is why many Soviet and foreign publications are devoted to this subject. A problem may be considered studied in detail, and that is the problem of a prismatic beam under the influence of an arbitrary transverse load and axial point loads imposed at the ends. This problem is known to be solved through application of elementary trigonometric and hyperbolic functions for compression or tension.

The computations become much more complicated when the longitudinal forces are distributed along the length of the beam. In such cases accurate solutions appear possible only for an extremely small number of variants of very simple transverse loads and for a more or less simple law of longitudinal load distribution. Even then the solutions involve special functions, which undoubtedly are difficult to use in engineering practice. For example, V. Makushin and N. Malinin applied Bessel functions to solve a prismatic beam under combined uniform longitudinal load and transverse load in the form of a single point load, a force couple, or a uniform load.

Investigations outlined in the present section give approximate solutions, which deviate slightly from the exact solutions; these approximate solutions are derived by the moment-operational method and may be applied to arbitrary transverse loads without using any special functions.

36. PRISMATIC BEAMS UNDER ARBITRARY TRANSVERSE LOADS AND CONSTANT AXIAL FORCES

Let a uniform beam (Fig. 36.1) of length l be loaded by a composite transverse load and by tensile forces S. Then the bending moment at any point X of the flexural curve in the deformed state consists of two components, e.g., the bending moment M_1 from the transverse load and the bending moment Sv from the longitudinal forces. The differential equation of the flexural curve in this case is given by

$$EI\frac{d^2v}{dx^2}=M_1+Sv \tag{36.1}$$

Substitution of

$$S=\alpha^2EI \tag{36.2}$$

into (36.1) gives the following equation for the flexural curve

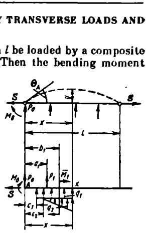

Fig. 36.1

$$EI\frac{d^2v}{dx^2}=M_1+\alpha^2EIv \tag{36.3}$$

Assuming for the first approximation that $\alpha=0$, i.e., by neglecting the longitudinal forces, we obtain

$$EI\frac{d^2v}{dx^2}=M_1$$

hence,

$$EI\frac{dv}{dx}=\frac{M_2}{2}$$

and

$$EIv=\frac{M_3}{3!} \tag{36.4}$$

Then from formula (36.3) we obtain

$$EI\frac{d^2v}{dx^2}=M_1+\alpha^2\frac{M_3}{3!} \tag{36.5}$$

the first and second integrals of which are given by

$$EI\frac{dv}{dx}=\frac{M_2}{2}+\alpha^2\frac{M_4}{4!} \tag{36.6}$$

and

$$EIv=\frac{M_3}{3!}+\alpha^2\frac{M_5}{5!} \tag{36.7}$$

Further successive approximations give the infinite moment series

$$EI\,\frac{dv}{dx}=\frac{M_2}{2}+\alpha^2\,\frac{M_4}{4!}+\alpha^4\,\frac{M_6}{6!}+\alpha^6\,\frac{M_8}{8!}+\dots \tag{36.8}$$

$$EIv=\frac{M_3}{3!}+\alpha^2\,\frac{M_5}{5!}+\alpha^4\,\frac{M_7}{7!}+\alpha^6\,\frac{M_9}{9!}+\dots \tag{36.9}$$

When the forces S are compressive (not tensile) formulas (36.8) and (36.9) remain true for comparatively small values of forces S, but the signs of the series alternate

$$EI_0\,\frac{dv}{dx}=\frac{M_2}{2}-\alpha^2\,\frac{M_4}{4!}+\alpha^4\,\frac{M_6}{6!}-\alpha^6\,\frac{M_8}{8!}+\dots \tag{36.10}$$

$$EIv=\frac{M_3}{3!}-\alpha^2\,\frac{M_5}{5!}+\alpha^4\,\frac{M_7}{7!}-\alpha^6\,\frac{M_9}{9!}+\dots \tag{36.11}$$

Formulas (36.8)-(36.11) may be expanded according to formula (III) of Sec. 8 for the usual composite transverse load, if initial angular and linear displacements are denoted by Θ_A and v_A.

For a beam under tensile and transverse forces

$$\begin{aligned}EI\,\frac{dv}{dx}&=EI\,\Theta_X=EIv_A\left[\alpha^2x+\alpha^4\,\frac{x^3}{3!}+\alpha^6\,\frac{x^5}{5!}+\dots\right]+\\&+EI\Theta_A\left[1+\alpha^2\,\frac{x^2}{2}+\alpha^4\,\frac{x^4}{4!}+\dots\right]+\\&+\sum\overline{M}_i\left[(x-a_i)+\alpha^2\,\frac{(x-a_i)^3}{3!}+\right.\\&\left.+\alpha^4\,\frac{(x-a_i)^5}{5\,!}+\dots\right]+\sum P_i\left[\frac{(x-b_i)^2}{2!}+\right.\\&\left.+\alpha^2\,\frac{(x-b_i)^4}{4!}+\alpha^4\,\frac{(x-b_i)^6}{6!}+\dots\right]+\\&+\sum q_i\left[\frac{(x-c_i)^3}{3!}+\alpha^2\,\frac{(x-c_i)^5}{5!}+\alpha^4\,\frac{(x-c_i)^7}{7!}+\dots\right]+\\&+\sum k_i\left[\frac{(x-d_i)^4}{4!}+\alpha^2\,\frac{(x-d_i)^6}{6!}+\alpha^4\,\frac{(x-d_i)^8}{8!}+\dots\right]\end{aligned} \tag{36.12}$$

$$\begin{aligned}EIv&=EIv_A\left[1+\alpha^2\,\frac{x^2}{2}+\alpha^4\,\frac{x^4}{4!}+\dots\right]+\\&+EI\Theta_A\left[x+\alpha^2\,\frac{x^3}{3!}+\alpha^4\,\frac{x^5}{5!}+\dots\right]+\\&+\sum\overline{M}_i\left[\frac{(x-a_i)^2}{2}+\alpha^2\,\frac{(x-a_i)^4}{4!}+\alpha^4\,\frac{(x-a_i)^6}{6!}+\dots\right]+\\&+\sum P_i\left[\frac{(x-b_i)^3}{3!}+\alpha^2\,\frac{(x-b_i)^5}{5!}+\alpha^4\,\frac{(x-b_i)^7}{7!}+\dots\right]+\\&+\sum q_i\left[\frac{(x-c_i)^4}{4!}+\alpha^2\,\frac{(x-c_i)^6}{6!}+\alpha^4\,\frac{(x-c_i)^3}{8!}+\dots\right]+\\&+\sum k_i\left[\frac{(x-d_i)^5}{5!}+\alpha^2\,\frac{(x-d_i)^7}{7!}+\alpha^4\,\frac{(x-d_i)^9}{9!}+\dots\right]\end{aligned} \tag{36.13}$$

For a beam under compressive and transverse forces

$$\begin{aligned}
EI\frac{dv}{dx} = EI\Theta_X = EIv_A\left[-\alpha^2 x + \alpha^4\frac{x^3}{3!} - \alpha^6\frac{x^5}{5!} + \dots\right] + \\
+ \sum EI\Theta_A\left[1 - \alpha^2\frac{x^2}{2} + \alpha^4\frac{x^4}{4!} - \alpha^6\frac{x^6}{6!} + \dots\right] + \\
+ \sum \overline{M}_i\left[(x-a_i) - \alpha^2\frac{(x-a_i)^3}{3!} + \alpha^4\frac{(x-a_i)^5}{5!} - \dots\right] + \\
+ \sum P_i\left[\frac{(x-b_i)^2}{2} - \alpha^2\frac{(x-b_i)^4}{4!} + \alpha^4\frac{(x-b_i)^6}{6!} - \dots\right] + \\
+ \sum q_i\left[\frac{(x-c_i)^3}{3!} - \alpha^2\frac{(x-c_i)^5}{5!} + \alpha^4\frac{(x-c_i)^7}{7!} - \dots\right] + \\
+ \sum k_i\left[\frac{(x-d_i)^4}{4!} - \alpha^2\frac{(x-d_i)^6}{6!} + \alpha^4\frac{(x-d_i)^8}{8!} - \dots\right] \qquad (36.14)
\end{aligned}$$

$$\begin{aligned}
EIv = EIv_A\left[1 - \alpha^2\frac{x^2}{2} + \alpha^4\frac{x^4}{4!} - \dots\right] + \\
+ EI\Theta_A\left[x - \alpha^2\frac{x^3}{3!} + \alpha^4\frac{x^5}{5!} - \dots\right] + \\
+ \sum \overline{M}_i\left[\frac{(x-a_i)^2}{2} - \alpha^2\frac{(x-a_i)^4}{4!} + \alpha^4\frac{(x-a_i)^6}{6!} - \dots\right] + \\
+ \sum P_i\left[\frac{(x-b_i)^3}{3!} - \alpha^2\frac{(x-b_i)^5}{5!} + \alpha^4\frac{(x-b_i)^7}{7!} - \dots\right] + \\
+ \sum q_i\left[\frac{(x-c_i)^4}{4!} - \alpha^2\frac{(x-c_i)^6}{6!} + \alpha^4\frac{(x-c_i)^8}{8!} - \dots\right] + \\
\sum k_i\left[\frac{(x-d_i)^5}{5!} - \alpha^2\frac{(x-d_i)^7}{7!} + \alpha^4\frac{(x-d_i)^9}{9!} - \dots\right] \qquad (36.15)
\end{aligned}$$

Formulas (36.12)-(36.15) may be written in a different way. To perform this, we will introduce the index "0" in terms, which allow for static loads (i.e., these terms will be written as M_3^0, M_5^0, etc.). Thus, only static loads will be taken into account by the moment symbols; kinematic quantities (initial values of angle and deflection) are excluded. Noting that the expressions in the first two brackets are infinite power series of expanded trigonometric and hyperbolic functions, we arrive at the following expressions for deflections, angles and (by further differentiation) bending moments and transverse forces at an arbitrary section of the flexural curve at a distance x from the origin.

For a beam under tensile and transverse forces

$$\begin{aligned}
EIv_X = EIv_A\cosh\alpha x + EI\Theta_A\frac{\sinh\alpha x}{\alpha} + \\
+ \left(\frac{M_3^0}{3!} + \alpha^2\frac{M_5^0}{5!} + \alpha^4\frac{M_7^0}{7!} + \dots\right) \qquad (36.16)
\end{aligned}$$

$$EI\Theta_X = \alpha EIv_A \sinh \alpha x + EI\Theta_A \cosh \alpha x + \\ + \left(\frac{M_2^0}{2!} + \alpha^2 \frac{M_4^0}{4!} + \alpha^4 \frac{M_6^0}{6!} + \dots \right) \tag{36.17}$$

$$M_X = \alpha^2 EIv_A \cosh \alpha x + \alpha EI\Theta_A \sinh \alpha x + \\ + \left(M_1^0 + \alpha^2 \frac{M_3^0}{3!} + \alpha^4 \frac{M_5^0}{5!} + \dots \right) \tag{36.18}$$

$$Q_X = \alpha^3 EIv_A \sinh \alpha x + \alpha^2 EI\Theta_A \cosh \alpha x + \\ + \left(M_0^0 + \alpha^2 \frac{M_2^0}{2} + \alpha^4 \frac{M_4^0}{4!} + \dots \right) \tag{36.19}$$

For a beam under compressive and transverse forces

$$EIv_X = EIv_A \cos \alpha x + EI\Theta_A \frac{\sin \alpha x}{\alpha} + \\ + \left(\frac{M_3^0}{3!} - \alpha^2 \frac{M_5^0}{5!} + \alpha^4 \frac{M_7^0}{7!} - \dots \right) \tag{36.20}$$

$$EI\Theta_X = -\alpha EIv_A \sin \alpha x + EI\Theta_A \cos \alpha x + \\ + \left(\frac{M_2^0}{2} - \alpha^2 \frac{M_4^0}{4!} + \alpha^4 \frac{M_6^0}{6!} - \dots \right) \tag{36.21}$$

$$M_X = -\alpha^2 EIv_A \cos \alpha x - \alpha EI\Theta_A \sin \alpha x + \\ + \left(M_1^0 - \alpha^2 \frac{M_3^0}{3!} + \alpha^4 \frac{M_5^0}{5!} - \dots \right) \tag{36.22}$$

$$Q_X = \alpha^3 EIv_A \sin \alpha x - \alpha^2 EI\Theta_A \cos \alpha x + \\ + \left(M_0^0 - \alpha^2 \frac{M_2^0}{2} + \alpha^4 \frac{M_4^0}{4!} - \dots \right) \tag{36.23}$$

Further detailed investigation is carried out for beam under combined compression and flexure. Taking various transverse loads into account (as shown in Fig. 36.1) but assuming forces S to be compressive, formulas (36.20)-(36.23) may be presented as

$$\begin{aligned} EIv_X &= EIv_A \cos \alpha x + EI\Theta_A \frac{\sin \alpha x}{\alpha} + \\ &+ \frac{1}{\alpha^2} \sum M_i \left[1 - \cos \alpha (x - a_i)\right] + \\ &+ \frac{1}{\alpha^3} \sum P_i \left[\alpha (x - b_i) - \sin \alpha (x - b_i)\right] + \\ &+ \frac{1}{\alpha^4} \sum q_i \left[\alpha^2 \frac{(x - c_i)^2}{2} - 1 + \cos \alpha (x - c_i)\right] + \\ &+ \frac{1}{\alpha^5} \sum k_i \left[\alpha^3 \frac{(x - d_i)^3}{3!} - \alpha (x - d_i) + \sin \alpha (x - d_i)\right] \end{aligned} \tag{36.24}$$

$$EI\Theta = -\alpha EIv_A \sin \alpha x + EI\Theta_A \cos \alpha x + \\ + \frac{1}{\alpha} \sum \overline{M}_i \sin \alpha (x - a_i) + \frac{1}{\alpha^2} \sum P_i [1 - \cos \alpha (x - b_i)] + \\ + \frac{1}{\alpha^3} \sum q_i [\alpha (x - c_i) - \sin \alpha (x - c_i)] + \ldots \qquad (36.25)$$

$$M_X = -\alpha^2 EIv_A \cos \alpha x - \alpha EI\Theta_A \sin \alpha x + \\ + \sum \overline{M}_i \cos \alpha (x - a_i) + \frac{1}{\alpha} \sum P_i \sin \alpha (x - b_i) + \\ + \frac{1}{\alpha^2} \sum q_i [1 - \cos \alpha (x - c_i)] + \ldots \qquad (36.26)$$

$$Q_X = \alpha^3 EIv_A \sin \alpha x - \alpha^2 EI\Theta_A \cos \alpha x - \alpha \sum \overline{M}_i \sin \alpha (x - a_i) + \\ + \sum P_i \cos \alpha (x - b_i) + \frac{1}{\alpha} \sum q_i \sin \alpha (x - c_i) + \ldots \qquad (36.27)$$

Thus, infinite moment series of the previous formulas may be represented by expanded expressions in terms of hyperbolic functions (for beams under combined tension and flexure) or trigonometric functions (for beams under combined compression and flexure) to obtain exact solutions.

In order to simplify formulas (36.24)-(36.27), as well as corresponding expressions for beams under combined tension and flexure, new generalized symbols will be introduced.

The following expression is called the *reduced moment* of the n-th order of all forces to the left of point X with abscissa x

$$M_n\alpha = \sum \alpha^n P_i (x - b_i)^n \qquad (36.28)$$

Further, the following expressions are the *reduced sine-moment* and *reduced cosine-moment* of forces to the left of point X:

$$\sin \alpha M = \sum P_i \sin \alpha (x - b_i) \qquad (36.29)$$

$$\cos \alpha M = \sum P_i \cos \alpha (x - b_i) \qquad (36.30)$$

Similarly, *hyperbolic sine-moment* and *hyperbolic cosine-moment* are

$$\sinh \alpha M = \sum P_i \sinh \alpha (x - b_i) \qquad (36.31)$$

$$\cosh \alpha M = \sum P_i \cosh \alpha (x - b_i) \qquad (36.32)$$

According to (36.28), the differentiation and integration rules may be formulated for the reduced moment of the n-th order as

$$\frac{d}{dx} M_n\alpha = n\alpha M_{n-1}\alpha \qquad (36.33)$$

$$\int M_n\alpha \, dx = \frac{1}{\alpha (n+1)} M_{n+1}\alpha \qquad (36.34)$$

Differentiation and integration of trigonometric moments is performed according to the following expressions derived from (36.29) and (36.30)

$$\frac{d}{dx} \sin \alpha M = \alpha \cos \alpha M \tag{36.35}$$

$$\int_0^x \sin \alpha M \, dx = \frac{1}{\alpha} \int_0^x \sin \alpha M d(\alpha x) = \frac{M_0 \alpha - \cos \alpha M}{\alpha} \tag{36.36}$$

$$\frac{d}{dx} \cos \alpha M = -\alpha \sin \alpha M \tag{36.37}$$

$$\int_0^x \cos \alpha M \, dx = \frac{\sin \alpha M}{\alpha} \tag{36.38}$$

Differentiation and integration rules for hyperbolic moments are similarly derived.

Introduction of trigonometric moments into (36.24)-(36.27) gives the following concise expressions

$$EIv_X = \frac{1}{\alpha^3} [M_1 \alpha - \sin \alpha M] \tag{36.39}$$

$$EI\Theta_X = \frac{1}{\alpha^2} [M_0 \alpha - \cos \alpha M] \tag{36.40}$$

$$M = \frac{1}{\alpha} \sin \alpha M \tag{36.41}$$

$$Q = \cos \alpha M \tag{36.42}$$

It should be noted that point loads indicate the order of any moment, including trigonometric and hyperbolic ones, i.e., the order of the term referring to point loads is equal to that of the moment determined by this series.

The general expression for the reduced moment of the n-th order of kinematic and static loads is given by

$$\begin{aligned} \frac{M_n \alpha}{n!} &= \alpha^3 EIv_A \frac{(\alpha x)^{n-3}}{(n-3)!} + \alpha^2 EI\Theta_A \frac{(\alpha x)^{n-2}}{(n-2)!} + \\ &+ \alpha \sum \overline{M}_i \frac{[\alpha (x-a_i)]^{n-1}}{(n-1)!} + \sum P_i \frac{[\alpha (x-b_i)]^n}{n!} + \\ &+ \frac{1}{\alpha} \sum q_i \frac{[\alpha (x-c_i)]^{n+1}}{(n+1)!} + \frac{1}{\alpha^2} \sum k_i \frac{[\alpha (x-d_i)]^{n+2}}{(n+2)!} \end{aligned} \tag{36.43}$$

General expressions for reduced trigonometric moments of kinematic and static loads are similarly composed

$$\begin{aligned}\sin\alpha M = {} & -\alpha^3 EIv_A\cos\alpha x-\alpha^2 EI\Theta_A\sin\alpha x+\\ & +\alpha\sum M_i\cos\alpha(x-a_i)+\sum P_i\sin\alpha(x-b_i)+\\ & +\frac{1}{\alpha}\sum q_i\left[1-\cos\alpha(x-c_i)\right]+\\ & +\frac{1}{\alpha^2}\sum k_i\left[\alpha(x-d_i)-\sin\alpha(x-d_i)\right]\end{aligned}\tag{36.44}$$

$$\begin{aligned}\cos\alpha M = {} & \alpha^3 EIv_A\sin\alpha x-\alpha^2 EI\Theta_A\cos\alpha x-\\ & -\alpha\sum \overline{M}_i\sin\alpha(x-a_i)+\sum P_i\cos\alpha(x-b_i)+\\ & +\frac{1}{\alpha}\sum q_i\sin\alpha(x-c_i)+\frac{1}{\alpha^2}\sum k_i\left[1-\cos\alpha(x-d_i)\right]\end{aligned}\tag{36.45}$$

Formulas (36.43)-(36.45) are the main subsidiary formulas, as well as (36.33)-(36.38). They are of the same significance in the theory of trigonometric and hyperbolic moments, as the main subsidiary formulas (I) and (III) in the theory of ordinary higher moments.

One of the main practical problems to be solved when applying the formulas derived in the present section is to decide on the number of terms in the moment series. Investigations undertaken by B. Lunin showed that sufficiently accurate solutions are obtained in a great number of problems, if only the initial angles and deflections as well as the first terms of moment series are taken into account. B. Lunin proved this by comparing his approximate solutions with the well-known solutions obtained by A. Van-der-Flit on the basis of theoretical preconditions in strength of materials. However, the accurate solutions of Van-der-Flit refer only to a limited range of loadings examined by the author; every new loading requires new calculations.

It will be shown below, that by limiting the moment series to two terms, quite accurate solutions may be obtained for various loads presented in Fig. 36.1.

Thus, we shall now proceed from the following types of formulas

$$EIv_X = EIv_A\cos\alpha x+EI\Theta_A\frac{\sin\alpha x}{\alpha}+\frac{M_3^0}{3!}-\alpha^2\frac{M_5^0}{5!}\tag{36.46}$$

$$EI\Theta_X = -\alpha EIv_A\sin\alpha x+EI\Theta_A\cos\alpha x+\frac{M_2^0}{2}-\alpha^2\frac{M_4^0}{4!}\tag{36.47}$$

$$M_X = -\alpha^2 EIv_A\cos\alpha x-\alpha EI\Theta_A\sin\alpha x+M_1^0-\alpha^2\frac{M_3^0}{3!}\tag{36.48}$$

$$Q_X = \alpha^3 EIv_A\sin\alpha x-\alpha^2 EI\Theta_A\cos\alpha x+M_0^0-\alpha^2\frac{M_2^0}{2}\tag{36.49}$$

in which the values of M_0^0, M_1^0, M_2^0, etc. must be determined from formula (I).

Examine a simple beam under the influence of longitudinal forces and a uniform transverse load (Fig. 36.2).

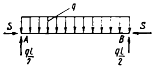

Fig. 36.2

Since the deflection at the right support is zero

$$\frac{M^0_{B^3}}{3!} - \alpha^2 \frac{M^0_{B^5}}{5!} + EI\Theta_A \cdot \frac{\sin \alpha l}{\alpha} = 0$$

hence,

$$EI\Theta_A = -\frac{\alpha}{\sin \alpha l}\left[\frac{M^0_{B^3}}{3!} - \alpha^2 \frac{M^0_{B^5}}{5!}\right] \qquad (36.50)$$

According to formula (I), higher moments of the given load are expressed by

$$\frac{M^0_{B^3}}{3!} = \frac{ql}{2} \cdot \frac{l^3}{3!} - \frac{ql^4}{4!} = \frac{ql^4}{24}$$

$$\frac{M^0_{B^5}}{5!} = \frac{ql}{2} \cdot \frac{l^5}{5!} - \frac{ql^6}{6!} = \frac{ql^6}{360}$$

Formula (36.46) gives the following expression for the deflection at mid-span (point C where $x = 1/2l$)

$$EIf = \frac{M^0_{C^3}}{3!} - \alpha^2 \frac{M^0_{C^5}}{5!} + \left(-\frac{M^0_{B^3}}{3!} + \alpha^2 \frac{M^0_{B^5}}{5!}\right) \frac{\sin \frac{\alpha l}{2}}{\sin l} \qquad (36.51)$$

where

$$\frac{M^0_{C^3}}{3!} = \frac{ql}{2}\left(\frac{l}{2}\right)^3 \frac{1}{3!} - \frac{q}{4!}\left(\frac{l}{2}\right)^4 = \frac{ql^4}{128}$$

$$\frac{M^0_{C^5}}{5!} = \frac{ql}{2}\left(\frac{l}{2}\right)^5 \frac{1}{5!} - \frac{q}{6!}\left(\frac{l}{2}\right)^6 = \frac{ql^4}{72 \cdot 128}$$

To simplify the calculations, a new subsidiary coefficient $u = \alpha l/2$ is introduced, which is then substituted into (36.51), thus resulting after collecting similar terms in the following expression

$$f = -\frac{5}{384} \cdot \frac{ql^4}{EI} k_1 \qquad (36.52)$$

where

$$k_1=\frac{3}{5}+\frac{u^2}{30}+\left(\frac{16}{5}-\frac{64u^2}{75}\right)\frac{\sin u}{\sin 2u} \tag{36.53}$$

while the exact solution gives

$$k_1'=\frac{\sec u-1-\frac{u^2}{2}}{\frac{5}{24}u^4} \tag{36.54}$$

Comparison of coefficients k_1 and k_1' for various values of u is given in Table 36.1, from which it may be observed, that the disagreement between the two coefficients is very small. For example, when $u = 60°$, this disagreement is about 5.5%. However,

Table 36.1

u, degrees	u^2, radians	For deflections		For moments	
		k_1	k_1'	k_2	k_2'
10	0.0304	1.00	1.00	1.013	1.013
20	0.1218	1.06	1.05	1.052	1.052
30	0.2742	1.13	1.12	1.127	1.128
40	0.4873	1.24	1.23	1.247	1.254
45	0.6169	1.34	1.32	1.332	1.342
60	1.0966	1.80	1.70	1.760	1.824

$$\left.\begin{aligned}&\frac{\alpha l}{2}=u;\quad S_{cr}=\frac{\pi^2EI}{l^2};\quad \alpha^2=\frac{S}{EI}\\ &S=\alpha^2EI=\frac{4u^2}{l^2}EI=\frac{u^2}{\left(\frac{1}{2}\pi\right)^2}S_{cr}\end{aligned}\right\} \tag{36.55}$$

The value of $u = 60°$ corresponds to an axial load which is equal to 44% of the critical load. Thus, formula (36.46), which contains two terms of the moment series, gives a sufficiently accurate solution for considerably large axial loads.

The largest possible bending moment which is at mid-span (due to symmetry) is determined by (36.48)

$$\max\ M=M^0_{C'}-\alpha^2\frac{M^0_{C^3}}{3!}-\frac{\alpha^2}{2}\sec u\left(-\frac{M^0_{B^3}}{3!}-\alpha^2\frac{M^0_{B^5}}{5!}\right) \tag{36.56}$$

Substitutions and transformations similar to those performed previously give

$$\max M=\frac{ql^2}{8}k_2 \tag{36.57}$$

where

$$k_2 = 1 - \frac{u^2}{4} + \left(\frac{2}{3}u^2 - \frac{8}{45}u^4\right)\sec u \tag{36.58}$$

while the exact solution for this coefficient is given by

$$k_2' = \frac{2(1-\cos u)}{u^2 \cos u} \tag{36.59}$$

Exact Solution in Terms of Trigonometric Moments

The "exact" solutions are obtained by formulas (36.24)-(36.27), which are somewhat rearranged to simplify their application to the various loading schemes applied to a beam.

If initial parameters are not introduced into the expressions represented by trigonometric moments, formulas (36.24)-(36.27) are written as

$$Q = \alpha^3 EIv_A \sin \alpha x - \alpha^2 EI\Theta_A \cos \alpha x + \cos \alpha M \tag{36.60}$$

$$M = -\alpha^2 EIv_A \cos \alpha x + \alpha EI\Theta_A \sin \alpha x + \frac{1}{\alpha} \sin \alpha M \tag{36.61}$$

$$EI\Theta_X = -\alpha EIv_A \sin \alpha x + EI\Theta_A \cos \alpha x + \frac{1}{\alpha^2}(\alpha M_0 - \cos \alpha M) \tag{36.62}$$

$$EIv_X = EIv_A \cos \alpha x + \frac{1}{\alpha} EI\Theta_A \sin \alpha x + \frac{1}{\alpha^3}(\alpha M_1 - \sin \alpha M) \tag{36.63}$$

where Θ_A, v_A = initial angular and linear displacements

M_0, M_1 = moments of zero and first orders of all loads to the left of section X with abscissa x with respect to vertical axis passing through point X

$\sin \alpha M$, $\cos \alpha M$ = sine- and cosine-moment of all loads (active and reactive) to the left of point X with respect to this point

For a compound load (Fig. 36.3)

$$M_0 = P + q(x-c) + \frac{k(x-d)^2}{2}$$

$$M_1 = \overline{M} + P(x-b) + \frac{q(x-c)^2}{2} + \frac{k(x-d)^3}{3!}$$

$$\sin \alpha M = \alpha \overline{M} \cos \alpha (x-a) + P \sin \alpha (x-b) +$$
$$+ \frac{1}{\alpha} q [1 - \cos \alpha (x-c)] + \frac{1}{\alpha^2} k [\alpha (x-d) - \sin \alpha (x-d)]$$

$$\cos \alpha M = -\alpha \overline{M} \sin \alpha (x-a) + P \cos \alpha (x-b) +$$
$$+ \frac{1}{\alpha} q \sin \alpha (x-c) + \frac{1}{\alpha^2} k [1 - \cos \alpha (x-d)]$$

If the transverse load applied to a beam consists of force couples, point loads, uniformly distributed and uniformly varying loads of various slopes (triangular loading areas), then all these loads are to be taken into account with appropriate subscripts within the sum sign.

$$M_0 = \sum P_i + \sum q_i (x - c_i) + \sum k_i \frac{(x-d_i)^2}{2} \tag{36.64}$$

$$M_1 = \sum \overline{M}_i + \sum P_i (x - b_i) + \frac{\sum q_i (x-c_i)^2}{2} + \\ + \sum k_i \frac{(x-d_i)^3}{3!} \tag{36.65}$$

$$\sin \alpha M = \alpha \sum \overline{M}_i \cos \alpha (x - a_i) + \sum P_i \sin \alpha (x - b_i) + \\ + \frac{1}{\alpha} \sum q_i [1 - \cos \alpha (x - c_i)] + \\ + \frac{1}{\alpha^2} \sum k_i [\alpha (x - d_i) - \sin \alpha (x - d_i)] \tag{36.66}$$

$$\cos \alpha M = -\alpha \sum \overline{M}_i \sin \alpha (x - a_i) + \sum P_i \cos \alpha (x - d_i) + \\ + \frac{1}{\alpha} \sum q_i \sin \alpha (x - c_i) + \frac{1}{\alpha^2} \sum k_i [1 - \cos \alpha (x - d_i)] \tag{36.67}$$

Practical use of formulas (36.60)-(36.63) requires calculations of trigonometric moments for various loading diagrams. Moments M_0

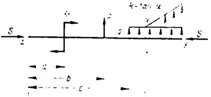

Fig. 36.3

and M_1 are shear force and bending moment at the given section, they are caused by transverse loads and are simple to determine. Sine-moments and cosine-moments are calculated for several types of transverse loads.

First, the relationship between trigonometric moments with respect to two parallel axes are established so that the required transition formulas are obtained for these two axes.

According to Fig. 36.4, the sine-moment of force P with respect to axis *1-1* is given by

$$\sin \alpha M' = P \sin \alpha x_1 \tag{36.68}$$

The sine-moment of the same force with respect to axis *2-2* is

$$\sin \alpha M'' = P \sin \alpha x_2 \tag{36.69}$$

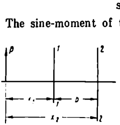

Fig. 36.4

Since $x_2 = x_1 + b$, hence

$$\sin \alpha M'' = P \sin \alpha (x_1 + b) = P \sin \alpha x_1 \cos \alpha b + P \cos \alpha x_1 \sin \alpha b$$

or

$$\sin \alpha M'' = \sin \alpha M' \cos \alpha b + \cos \alpha M' \sin \alpha b \tag{36.70}$$

This relationship may be written as

$$\sin \alpha M'' = \sin \alpha (M' + b) \tag{36.71}$$

Similarly, the transition formula for the cosine-moment is

$$\cos \alpha M'' = \cos \alpha M' \cos \alpha b - \sin \alpha M' \sin \alpha b \tag{36.72}$$

or

$$\cos \alpha M'' = \cos \alpha (M' + b) \tag{36.73}$$

Arrangement of terms in (36.64) and (36.65) makes it possible to apply formulas (36.66) and (36.67) to any particular loading diagram.

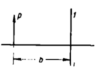

Fig. 36.5

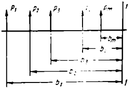

Fig. 36.6

Consider trigonometric moments for the most popular loading diagrams.

Loading diagram No. 1: a single point load (Fig. 36.5)

$$\sin \alpha M' = P \sin \alpha b \tag{36.74}$$

$$\cos \alpha M' = P \cos \alpha b \tag{36.75}$$

Loading diagram No. 2: a system of m arbitrarily arranged point forces (Fig. 36.6)

$$\sin \alpha M' = P_1 \sin \alpha b_1 + P_2 \sin \alpha b_2 + \dots + P_m \sin \alpha b_m = \sum_{i=1}^{m} P_i \sin \alpha b_i \tag{36.76}$$

$$\cos\alpha M' = P_1\cos\alpha b_1 + P_2\cos\alpha b_2 + \ldots +$$
$$+ P_m\cos\alpha b_m = \sum_{i=1}^{m} P_i\cos\alpha b_i \qquad (36.77)$$

Loading diagram No. 3: a system of m equal forces applied at equal distances from each other and from the moment point (Fig. 36.7)

$$\sin\alpha M' = P\sum_{i=1}^{m}\sin\alpha b_i \qquad (36.78)$$

$$\cos\alpha M' = P\sum_{i=1}^{m}\cos\alpha b_i \qquad (36.79)$$

Using the summation formulas of the trigonometric functions, we obtain

$$\sin\alpha M' = P\sin\frac{m+1}{2}\alpha b\sin\frac{m\alpha b}{2}\operatorname{cosec}\frac{\alpha b}{2} \qquad (36.80)$$

$$\cos\alpha M' = P\cos\frac{m+1}{2}\alpha b\sin\frac{m\alpha b}{2}\operatorname{cosec}\frac{\alpha b}{2} \qquad (36.81)$$

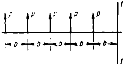

Fig. 36.7

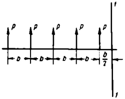

Fig. 36.8

Loading diagram No. 4: a system of m equal point forces applied at equal distances from each other, while the distance from the last force to the moment point is equal to $\frac{b}{2}$ (Fig. 36.8)

$$\sin\alpha M' = P\sum_{i=1}^{m}\sin\frac{2i-1}{2}\alpha b \qquad (36.82)$$

$$\cos\alpha M' = P\sum_{i=1}^{m}\cos\frac{2i-1}{2}\alpha b \qquad (36.83)$$

Loading diagram No. 5: uniformly distributed load (Fig. 36.9). The sine-moment is determined by a sum of elementary forces qdx

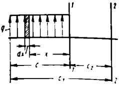

Fig. 36.9

$$\sin\alpha M' = \int_0^c q\sin\alpha x\,dx =$$

$$= \frac{q}{\alpha}(1-\cos\alpha c) \qquad (36.84)$$

Similarly, the cosine-moment is

$$\cos\alpha M' = \int_0^c q\cos\alpha x\,dx = \frac{q}{\alpha}\sin\alpha c \qquad (36.85)$$

Sine-moments and cosine-moments with respect to axis *2-2* are determined according to transition formulas (36.70) and (36.72)

$$\sin\alpha M'' = \sin\alpha M'\cos\alpha c_2 + \cos\alpha M'\sin\alpha c_2 =$$

$$= \frac{q}{\alpha}(1-\cos\alpha c)\cos\alpha c_2 + \frac{q}{\alpha}\sin\alpha c\sin\alpha c_2 =$$

$$= \frac{q}{\alpha}(\cos\alpha c_2 - \cos\alpha c_1) \qquad (36.86)$$

This formula may be derived by determining the unknown moment as a difference between moments from loads qc_1 and qc_2, respectively. Cosine-moments are similarly determined. Thus,

$$\sin\alpha M'' = \frac{q}{\alpha}(\cos\alpha c_2 - \cos\alpha c_1) \qquad (36.87)$$

$$\cos\alpha M'' = \frac{q}{\alpha}(\sin\alpha c_1 - \sin\alpha c_2) \qquad (36.88)$$

Loading diagram No.6: an increasing triangular load (Fig. 36.10). Taking an elementary loading area at distance x from axis *1-1* and noting that $y = \frac{p}{d}(d-x)$, we find

$$\sin\alpha M' = \int_0^d y\sin\alpha x\,dx = \frac{p}{\alpha^2 d}(\alpha d - \sin\alpha d) \qquad (36.89)$$

Similarly

$$\cos\alpha M' = \frac{p}{\alpha^2 d}(1-\cos\alpha d) \qquad (36.90)$$

Moments with respect to axis *2-2* are determined by transition formulas

$$\sin\alpha M'' = \frac{p}{\alpha^2 d}(\sin\alpha d_2 - \sin\alpha d_1) + \frac{p}{\alpha}\cos\alpha d_2 \qquad (36.91)$$

$$\cos\alpha M'' = \frac{p}{\alpha^2 d}(\cos\alpha d_2 - \cos\alpha d_1) - \frac{p}{\alpha}\sin\alpha d_2 \qquad (36.92)$$

Loading diagram No. 7: a decreasing triangular load (Fig. 36.11). Previous formulas may be applied, if the given loading area is determined as the difference between the rectangle *abcd* and the triangle *bcd*

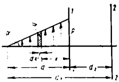

Fig. 36.10

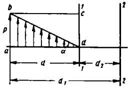

Fig. 36.11

$$\sin\alpha M' = \frac{p}{\alpha}(1 - \cos\alpha d) - \frac{p}{\alpha^2 d}(\alpha d - \sin\alpha d) \qquad (36.93)$$

or

$$\sin\alpha M' = \frac{p}{\alpha^2 d}(\sin\alpha d - \alpha d\cos\alpha d) \qquad (36.94)$$

and

$$\cos\alpha M' = \frac{p}{\alpha^2 d}(\cos\alpha d - \alpha d\sin\alpha d) \qquad (36.95)$$

Passing to the parallel axis *2-2*

$$\sin\alpha M'' = \frac{p}{\alpha^2 d}(\sin\alpha d_2) + \frac{p}{\alpha}\sin\alpha d_1 \qquad (36.96)$$

$$\cos\alpha M'' = \frac{p}{\alpha^2 d}(\cos\alpha d_1 + \cos\alpha d_2) + \frac{p}{\alpha}\sin\alpha d_1 \qquad (36.97)$$

Loading diagram No. 8: a symmetrical triangular load (Fig. 36.12). Expressions for the sine-moments and cosine-moments with respect to axes *1-1* and *2-2* are determined by formulas derived for loading diagrams No. 6 and No. 7

$$\sin\alpha M' = \frac{p}{\alpha^2 d}(\sin\alpha d - \sin 2\alpha d) + \frac{p}{\alpha}\cos\alpha d +$$
$$+ \frac{p}{\alpha^2 d}(\sin\alpha d - \alpha d\cos\alpha d) = \frac{2p}{\alpha^2 d}\sin\alpha d(1 - \cos\alpha d)$$

hence,

$$\sin\alpha M' = \frac{p}{\alpha^2 d}(2\sin\alpha d - \sin 2\alpha d) \qquad (36.98)$$

$$\cos\alpha M' = \frac{p}{\alpha^2 d}(2\cos\alpha d - \cos 2\alpha d) \qquad (36.99)$$

For axis *2-2*

$$\sin\alpha M'' = \frac{p}{\alpha^2 d}[\sin\alpha(d+d_2) - \sin\alpha d_1] + \frac{p}{\alpha}\cos\alpha(d+d_2) +$$

$$+\frac{p}{\alpha^2 d}[\sin\alpha(d+d_2) + \sin\alpha d_2] - \frac{p}{\alpha}\cos\alpha(d+d_2) =$$

$$= \frac{p}{\alpha^2 d}[2\sin\alpha(d+d_2) - \sin\alpha d_1 + \sin\alpha d_2] \qquad (36.100)$$

$$\cos\alpha M'' = \frac{p}{\alpha^2 d}[2\cos\alpha(d+d_2) - \cos\alpha d_1 + \cos\alpha d_2] \qquad (36.101)$$

Loading diagram No. 9: an asymmetrical triangular loading area (Fig. 36.13).

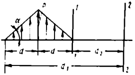

Fig. 36.12

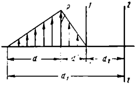

Fig. 36.13

According to formulas derived for loading diagrams No. 6 and No. 7 we find

$$\sin\alpha M' = \frac{p}{\alpha^2 d}[\sin\alpha d' - \sin\alpha(d+d')] + \frac{p}{\alpha^2 d'}\sin\alpha d' \quad (36.102)$$

$$\cos\alpha M' = \frac{p}{\alpha^2 d}[\cos\alpha d' - \cos\alpha(d+d')] + \frac{p}{\alpha^2 d}\cos\alpha d' \quad (36.103)$$

With respect to axis *2-2*

$$\sin\alpha M'' = \frac{p}{\alpha^2 d}[\sin\alpha(d'+d_2) - \sin\alpha d_1] +$$

$$+\frac{p}{\alpha^2 d'}[\sin\alpha(d'+d_2) + \sin\alpha d_2] \qquad (36.104)$$

$$\cos\alpha M'' = \frac{p}{\alpha^2 d}[\cos\alpha(d'+d_2) - \cos\alpha d_1] +$$

$$+\frac{p}{\alpha^2 d'}[\cos\alpha(d'+d_2) + \cos\alpha d_2] \qquad (36.105)$$

Loading diagram No. 10: a trapezoidal load (Fig. 36.14)

$$\sin \alpha M' = \frac{p_1 + p_2}{\alpha} - \frac{p_1}{\alpha}\cos \alpha d - \frac{p_2}{\alpha^2 d}\sin \alpha d \qquad (36.106)$$

$$\cos \alpha M' = \frac{p_1}{\alpha}\sin \alpha d - \frac{p_2}{\alpha^2 d}\cos \alpha d \qquad (36.107)$$

$$\sin \alpha M'' = \frac{p_1 + p_2}{\alpha}\cos \alpha d_2 - \frac{p_1}{\alpha}\cos \alpha d_1 + $$

$$+ \frac{p_2}{\alpha^2 d}(\sin \alpha d_2 - \sin \alpha d_1) \qquad (36.108)$$

$$\cos \alpha M'' = \frac{p_1}{\alpha}\sin \alpha d_1 + \frac{p_2}{\alpha^2 d}(\cos \alpha d_2 - \cos \alpha d_1) \qquad (36.109)$$

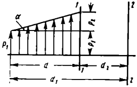

Fig. 36.14

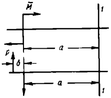

Fig. 36.15

Loading diagram No. 11: a force couple (Fig. 36.15). Denoting $\bar{M} = P\delta$

$$\sin \alpha M' = P \sin \alpha (a + \delta) - P \sin \alpha a =$$

$$= P \sin \alpha d (\cos \alpha\delta - 1) + P \cos \alpha a \sin \alpha\delta$$

Assuming $\delta \to 0$ and $P \to \infty$, so that $P\delta = \bar{M}$

$$\sin \alpha M' = P \cos \alpha \ \ a\alpha\delta = \alpha \bar{M} \cos \alpha a \qquad (36.110)$$

$$\cos \alpha M' = -\alpha \bar{M} \sin \alpha a \qquad (36.111)$$

Formulas derived for the loading diagrams examined above make it possible to use the general solutions given by (36.60)-(36.63) without expanding the trigonometric moments according to (36.64)-(36.65); these general formulas may be applied to determine initial parameters, unknown stresses and displacements caused by various transverse loads.

37. ROTATING ROD OF CONSTANT RIGIDITY UNDER COMPRESSION AND FLEXURE

The author of this book noted that mechanical properties of materials improve more rapidly than their physical properties. As a result, components of heightened flexibility become more widely applied. Evaluation of such components for simultaneous action of bending moment and axial compression appears to be not accurate enough and practically inadequate when performed on the basis of the initial geometry. In this connection, calculations of the steady strength become increasingly important, i.e., computation of the deformed state of the system with the influence of displacements upon the stress taken into account. In addition, increasing speeds of moving members increases the influence of inertia forces, which cause an additional loading and must be also taken into consideration.

Such problems may be solved by using moments of higher order and the moment-operational method.

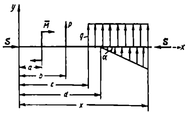

Fig. 37.1

In the present section a compression-curved rod of constant rigidity under arbitrary transverse load and constant longitudinal force is taken as an example of application of the moment-operational method to determine approximate displacements taking into account inertia forces arising as the rod rotates about its axis.

Differential equation of the flexural curve is given by

$$EI\frac{d^4v}{dx^4} = M_{-1} - \alpha^2 EI\frac{d^2v}{dx^2} + \beta^2 EIv \tag{37.1}$$

where

$$\alpha^2 = \frac{S}{EI}\,; \quad \beta^2 = \frac{m\omega^2}{EI}$$

m = distributed mass

$M_{-1} = q + k(x - d)$ = moment of minus first order of the loading or the intensity of distributed load

The compound transverse load shown in Fig. 37.1 by several loading factors of various types may just as well be composed of an arbitrary number of loads. Application of the moment-operational method makes it possible to derive a general solution, which does not depend on any particular transverse loading. The latter is taken into account only after the final formulas are obtained when particular higher moments are to be expanded.

The moment of the n-th order of static and kinematic loads is determined by the following expression

$$\frac{M_n}{n!} = EIv_0\frac{x^{n-3}}{(n-3)!} + EI\Theta_0\frac{x^{n-2}}{(n-2)!} + \sum M_i\frac{(x-a_i)^{n-1}}{(n-1)!} + \\ + \sum P_i\frac{(x-b_i)^n}{n!} + \sum q_i\frac{(x-c_i)^{n+1}}{(n+1)!} + \sum k_i\frac{(x-d_i)^{n+2}}{(n+2)!} \tag{37.2}$$

In this formula i designates differently arranged loads of the same type, and v_0 and Θ_0 are the initial kinematic parameters.

Equation (37.1) is solved by the method of successive approximations. Assuming, as usual, that initially $\alpha = 0$ and $\beta = 0$, we obtain the condition of pure flexure:

$$EI\frac{d^4v}{dx^4} = M_{-1}$$

Noting that

$$\int \frac{M_n}{n!}\,dx = \frac{M_{n+1}}{(n+1)!}$$

and

$$\frac{d}{dx}\left(\frac{M_n}{n!}\right) = \frac{M_{n-1}}{(n-1)!}$$

the third and fourth integrals give in the zero approximation the angle and deflection, respectively

$$EI\frac{dv}{dx} = \frac{M_2}{2!}\,;\ EIv = \frac{M_3}{3!}$$

Substitution of this approximation into the right-hand part of (37.1) for the expressions containing v results in a new differential equation, integration of which gives the next approximation.

For simplicity of presentation, only the deviations of two successive approximations will be determined in order to establish the regularity in the structure of the approximations and to derive the general expression for the n-th approximation. Thus, we have avoided the cumbersome calculations associated with the derivation of expanded approximations which would make this analysis difficult.

Now let us examine the following equation

$$EI\frac{d^4v_i}{dx^4} = -\alpha^2 EI\frac{d^2v_i}{dx^2} + \beta^2 EIv_i \tag{37.3}$$

the right-hand part of which contains only those terms which were neglected in the first approximation. When solving this equation we determine the correction v_i corresponding to the n-th approximation.

Equation (37.3) on the basis of the previous expression may be presented as

$$EI\frac{d^4v_1}{dx^4} = -\alpha^2 M_1 + \beta^2\frac{M^3}{3!} \tag{37.4}$$

From this we obtain the correction v_1 for the first approximation

$$EI\frac{dv_1}{dx} = -\alpha^2\frac{M_4}{4!} + \beta^2\frac{M_6}{6!} \tag{37.5}$$

$$EIv_1 = -\alpha^2\frac{M_5}{5!} + \beta^2\frac{M_7}{7!} \tag{37.6}$$

Taking into account the obtained correction to the zero approximation we obtain the first approximation, then equation (37.3) gives a more accurate formula

$$EI\frac{d^4v_2}{dx^4} = \alpha^4\frac{M_3}{3!} - 2\alpha^2\beta^2\frac{M_5}{5!} + \beta^4\frac{M_7}{7!} \tag{37.7}$$

hence,

$$EI\frac{dv_2}{dx} = \alpha^4\frac{M_6}{6!} - 2\alpha^2\beta^2\frac{M_8}{8!} + \beta^4\frac{M_{10}}{10!} \tag{37.8}$$

$$EIv_2 = \alpha^4\frac{M_7}{7!} - 2\alpha^2\beta^2\frac{M_9}{9!} + \beta^4\frac{M_{11}}{11!} \tag{37.9}$$

Further substitution of these expressions into equation (37.3) gives a still more accurate formula

$$EI\frac{d^4v_3}{dx^4} = -\alpha^6\frac{M_5}{5!} + 3\alpha^4\beta^2\frac{M_7}{7!} - 3\alpha^2\beta^4\frac{M_9}{9!} + \beta^6\frac{M_{11}}{11!} \tag{37.10}$$

the integration of which gives v_3, i.e., the correction for the third approximation

$$EI\frac{dv_3}{dx} = -\alpha^6\frac{M_8}{8!} + 3\alpha^4\beta^2\frac{M_{10}}{10!} - 3\alpha^2\beta^4\frac{M_{12}}{12!} + \beta^6\frac{M_{14}}{14!} \tag{37.11}$$

$$EIv_3 = -\alpha^6\frac{M_9}{9!} + 3\alpha^4\beta^2\frac{M_{11}}{11!} - 3\alpha^2\beta^4\frac{M_{13}}{13!} + \beta^6\frac{M_{15}}{15!} \tag{37.12}$$

Similarly, v_4, the correction for the fourth approximation, is obtained

$$EI\frac{dv_4}{dx} = \alpha^8\frac{M_{10}}{10!} - 4\alpha^6\beta^2\frac{M_{12}}{12!} + 6\alpha^4\beta^4\frac{M_{14}}{14!} - 4\alpha^2\beta^6\frac{M_{16}}{16!} + \beta^8\frac{M_{18}}{18!} \tag{37.13}$$

$$EIv_4 = \alpha^8\frac{M_{11}}{11!} - 4\alpha^6\beta^2\frac{M_{13}}{13!} + 6\alpha^4\beta^4\frac{M_{15}}{15!} - 4\alpha^2\beta^6\frac{M_{17}}{17!} + \beta^8\frac{M_{19}}{19!} \tag{37.14}$$

These four corrections, (37.5)-(37.14), are sufficient to establish their regularity and to derive a formula for the correction of the n-th approximation.

The correction, which should be added to the $(n - 1)$-th approximation to obtain the n-th approximation of the angle, has the form

$$EI\frac{dv_n}{dx} = EI\Theta_n = \sum_{i=0}^{n} (-1)^{n+1}\binom{i}{n}\alpha^{2(n-1)}\beta^{2i}\frac{M_{2n+2i+2}}{(2n+2i+2)!} \tag{37.15}$$

The correction, which should be added to the $(n - 1)$-th approximation to obtain the n-th approximation of the deflection, has the form

$$EIv_n = \sum_{i=0}^{n} (-1)^{n+1}\binom{i}{n}\alpha^{2(n-1)}\beta^{2i}\frac{M_{2n+2i+3}}{(2n+2i+3)!} \tag{37.16}$$

Thus, an expression for the angular displacement in an arbitrary section

$$EI\Theta_x = \frac{M_2}{2!} + EI(\Theta_1 + \Theta_2 + \Theta_3 + \ldots) \qquad (37.17)$$

may be now written as

$$EI\Theta_x = \frac{M_2}{2!} + \sum_{i=0}^{1} (-1)^{n+1} \binom{i}{1} \alpha^{2-2i}\beta^{2i} \frac{M_{5+2i}}{(5+2i)!} + $$

$$+ \sum_{i=0}^{2} (-1)^{2+i} \binom{i}{n} \alpha^{2(2-i)}\beta^{2i} \frac{M_{7+2i}}{(7+2i)!} + \ldots$$

or more briefly as

$$EI\Theta_x = \sum_{n=0}^{\infty} \sum_{i=0}^{i=n} (-1)^{n+i} \binom{i}{n} \alpha^{2(n-i)}\beta^{2i} \frac{M_{2n+2i+2}}{(2n+2i+2)!} \qquad (37.18)$$

Similarly, the following expression for the deflection in an arbitrary section of the rod

$$EIv_x = \frac{M_3}{3!} + EI(v_1 + v_2 + v_3 + \ldots) \qquad (37.19)$$

may be on the basis of (37.16) transformed to

$$EIv_x = \sum_{n=0}^{\infty} \sum_{i=0}^{n} (-1)^{n+i} \binom{i}{n} \alpha^{2(n-i)}\beta^{2i} \frac{M_{2n+2i+3}}{(2n+2i+3)!} \qquad (37.20)$$

Thus, as a result of successive iterations, the general solution is expressed in the form of a power series.

Symbols of moments included in these formulas may be expanded to correspond to the particular loading according to formulas and diagrams of Sec. 7.

In order to estimate the accuracy of any given approximation, the last remaining term of the infinite moment series should be evaluated in each particular case, because the classical solution of a differential equation and its approximate solution by the moment-operational method are closely connected: the approximate solution in terms of moment symbols, obtained by iteration, as well as Pikar's solution of ordinary differential equations, is Taylor-Maclourin's series.

Formulas (37.18) and (37.20) are generalized expressions for displacements in rods under combined loading. When $\beta = 0$, they transform into the usual moment series

$$EI\Theta_x = \frac{M_2}{2!} - \alpha^2\frac{M_4}{4!} + \alpha^4\frac{M_6}{6!} - \alpha^6\frac{M_8}{8!} + \dots$$

$$EIv_x = \frac{M_3}{3!} - \alpha^2\frac{M_5}{5!} + \alpha^4\frac{M_7}{7!} - \alpha^6\frac{M_9}{9!} + \dots$$

which are used for approximate rigidity calculations.

38. PRISMATIC BEAM UNDER ARBITRARY TRANSVERSE LOAD AND UNIFORMLY DISTRIBUTED AXIAL FORCES

Let a beam of constant rigidity EI be loaded by a distributed axial compressive load of intensity p applied along the total length of the beam, and by an arbitrary transverse compound load consisting of point force couples, point forces, uniformly distributed loads and uniformly varying (decreasing or increasing) distributed loads (only one load of each type is shown in Fig. 38.1).

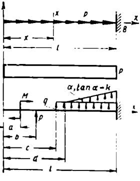

Fig. 38.1

The differential equation of the flexural curve for this particular case is given by

$$EI\frac{d^3v}{dx^3} + px\frac{dv}{dx} = M_0 \qquad (38.1)$$

where $px\frac{dv}{dx}$ = axial force at any arbitrary section X with abscissa x

M_0 = zero order moment of the left forces or transverse force of the transverse load

v = deflection of the axis of the beam measured upwards

EI = rigidity of the beam

Denoting

$$p = \beta EI \qquad (38.2)$$

we obtain

$$EI\frac{d^3v}{dx^3} = M_0 - \beta xEI\frac{dv}{dx} \qquad (38.3)$$

Assuming for the first approximation that $\beta = 0$, i.e., by neglecting the axial forces, we have

$$EI\frac{d^3v}{dx^3} = M_0$$

which after integration gives

$$EI\frac{dv}{dx}=\frac{M_2}{2!} \tag{38.4}$$

$$EIv=\frac{M_3}{3!} \tag{38.5}$$

Further approximations will be derived from (38.3). in this case M_0 will be disregarded, because its influence is allowed for in expressions (38.4) and (38.5) derived for $\frac{dv}{dx}$ and v. In other words, the more accurate successive approximations will be determined by including in expressions for $\frac{dv}{dx}$ and v only those supplementary terms by which any considered approximation is distinguished from the previous one. Thus, for the given case

$$EI\frac{d^3v}{dx^3}=-\beta xEI\frac{dv}{dx} \tag{38.6}$$

The second and third integrals are expressed by formulas of Sec. 20 for particular values of the index.

Having the subsidiary formulas, we return to (38.6) and substitute into it (for the first approximation) the value $EI\frac{dv}{dx}$ from formulas (38.4). Then we obtain

$$EI\frac{d^3v}{dx^3}=-\beta x\frac{M_2}{2} \tag{38.7}$$

For the second and third integrations of the obtained expression we use the subsidiary formulas, and when $n=2$ we find

$$EI\frac{dv}{dx}=-\beta\left(x\frac{M_4}{4!}-2\frac{M_5}{5!}\right) \tag{38.8}$$

and

$$EIv=-\beta\left(x\frac{M_5}{5!}-3\frac{M_6}{6!}\right) \tag{38.9}$$

Similarly, the next approximation is given by

$$EI\frac{d^3v}{dx^3}=2\beta^2\left(\frac{x^2}{2}\cdot\frac{M_4}{4!}-x\frac{M_5}{5!}\right) \tag{38.10}$$

For the second integral of the right-hand part of this expression we apply formulas of Sec. 20

$$EI\frac{dv}{dx}=2\beta^2\left[\left(\frac{x^2}{2}\cdot\frac{M_6}{6!}-2x\frac{M_7}{7!}+3\frac{M_8}{8!}\right)-\right.$$
$$\left.-\left(x\frac{M_7}{7!}-2\frac{M_8}{8!}\right)\right]=2\beta^2\left(\frac{x^2}{2}\cdot\frac{M_6}{6!}-3x\frac{M_7}{7!}+5\frac{M_8}{8!}\right) \tag{38.11}$$

$$EIv = 2\beta^2\left[\left(\frac{x^2}{2}\cdot\frac{M_7}{7!} - 3x\frac{M_8}{8!} + 6\frac{M_9}{9!}\right) - \right.$$

$$\left. - \left(x\frac{M_8}{8!} - 3\frac{M_9}{9!}\right)\right] = 2\beta^2\left(\frac{x^2}{2}\cdot\frac{M_7}{7!} - 4x\frac{M_8}{8!} + 9\frac{M_9}{9!}\right) \quad (38.12)$$

Similarly,

$$EI\frac{d^3v}{dx^3} = -2\beta^3\left[3\frac{x^3}{3!}\cdot\frac{M_6}{6!} - 6\frac{x^2}{2}\cdot\frac{M_7}{7!} + 5x\frac{M_8}{8!}\right] \quad (38.13)$$

Integration of this equation is performed by applying the same subsidiary formulas, and assuming $n = 6$, 7 and 8. After integration and collecting similar terms

$$EI\frac{dv}{dx} = -2\beta^3\left[3\frac{x^3}{3!}\cdot\frac{M_8}{8!} - 12\frac{x^2}{2!}\cdot\frac{M_9}{9!} + 26x\frac{M_{10}}{10!} - 40\frac{M_{11}}{11!}\right] \quad (38.14)$$

and

$$EIv = -2\beta^3\left[3\frac{x^3}{3!}\cdot\frac{M_9}{9!} - 15\frac{x^2}{2!}\cdot\frac{M_{10}}{10!} + 41x\frac{M_{11}}{11!} - 81\frac{M_{12}}{12!}\right] \quad (38.15)$$

Combining the obtained formulas into one general expression, we get

$$EI\frac{dv}{dx} = \frac{M_2}{2} - \beta\left(x\frac{M_4}{4!} - 2\frac{M_5}{5!}\right) + \beta^2\left(x^2\frac{M_6}{6!} - 6x\frac{M_7}{7!} + 10\frac{M_8}{8!}\right) -$$

$$- \beta^3\left(x^3\frac{M_8}{8!} - 12x^2\frac{M_9}{9!} + 52x\frac{M_{10}}{10!} - 80\frac{M_{11}}{11!}\right) + \ldots \quad (38.16)$$

$$EIv = \frac{M_3}{3!} - \beta\left(x\frac{M_5}{5!} - 3\frac{M_6}{6!}\right) + \beta^2\left(x^2\frac{M_7}{7!} - 8x\frac{M_8}{8!} + 18\frac{M_9}{9!}\right) -$$

$$- \beta^3\left(x^3\frac{M_9}{9!} - 15x^2\frac{M_{10}}{10!} + 82x\frac{M_{11}}{11!} - 162\frac{M_{12}}{12!}\right) + \ldots \quad (38.17)$$

The moment series obtained from the integration are rapidly converging, alternating, monotonously decreasing series. They converge so rapidly, that two terms of a series are usually sufficient for all practical engineering applications. Because of this, for the further calculations of angles and deflections we shall only use the following formulas

$$EI\frac{dv}{dx} = \frac{M_2}{2} - \beta\left(x\frac{M_4}{4!} - 2\frac{M_5}{5!}\right) \quad (38.18)$$

$$EIv = \frac{M_3}{3!} - \beta\left(x\frac{M_5}{5!} - 3\frac{M_6}{6!}\right) \quad (38.19)$$

These formulas will be used in the rigidity design of a beam; for strength designing, however, formulas for bending moments and transverse forces are required. These formulas are obtained by taking the first and second derivatives of (38.18)

$$Q = M_0 - \beta x \frac{M_2}{2} \tag{38.20}$$

$$M = M_1 - \beta \left(x \frac{M_3}{3!} - \frac{M_4}{4!} \right) \tag{38.21}$$

The formulas make it possible to solve a beam carrying an arbitrary compound transversal load, whereas without the symbolics of higher order moments up to the present time solutions have been possible only for a limited number of simple cases, and this by assuming that axial load is distributed uniformly along the axis of the beam.

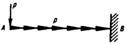

Fig. 38.2

We shall show that the approximate solution deviates only a little from the exact solution, as can be seen from the comparison of the exact solution and the approximate solution given by two terms of the infinite moment series. Let a beam be loaded by uniformly distributed axial forces and by a point load applied to the free end (Fig. 38.2). Expanding the symbolic moments in formula (38.19) and taking into account the initial angle (Θ_A) and the initial deflection (v_A), on the basis of formula (III) we obtain

$$\begin{aligned} EIv_x &= \frac{Px^3}{3!} - \beta \left(x \frac{Px^5}{5!} - 3P \frac{x^6}{6!} \right) + \\ &+ EI\Theta_A \left[x - \beta \left(x \frac{x^3}{3!} - 3 \frac{x^4}{4!} \right) \right] + \\ &+ EIv_A \left[1 - \beta \left(x \frac{x^2}{2} - 3 \frac{x^3}{3!} \right) \right] = \\ &= P \left(\frac{x^3}{3!} - 3\beta \frac{x^6}{6!} \right) + EI\Theta_A \left(x - \beta \frac{x^4}{4!} \right) + EIv_A \end{aligned}$$

As the deflection is zero at the right support ($v_B = 0$, when $x = l$), then

$$EIv_B = P \left(\frac{l^3}{3!} - 3\beta \frac{l^6}{6!} \right) + EI\Theta_A \left(l - \beta \frac{l^4}{4!} \right) + EIv_A = 0 \tag{38.22}$$

Similarly, since the angle is zero at the fixed end, according to (38.18), when $x = l$ we have

$$EI\Theta_B = P \left(\frac{l^2}{2} - 3\beta \frac{l^5}{5!} \right) + EI\Theta_A \left(l - \beta \frac{l^3}{3!} \right) = 0 \tag{38.23}$$

It follows from (38.22) and (38.23), that

$$EI\Theta_A = -\frac{Pl^2}{2} k_1 \tag{38.24}$$

where

$$k_1 = \frac{60 - 3\beta l^3}{60 - 10\beta l^3} \tag{38.25}$$

and

$$EIv_A = \frac{Pl^3}{3!} k_2 \tag{38.26}$$

where

$$k_2 = \frac{960 - 40\beta l^3 + (\beta l^3)^2}{960 - 160\beta l^3} \tag{38.27}$$

Designating $\beta l^3 = \frac{9}{4}\eta^2$ to simplify the comparison with the exact solutions we obtain

$$k_2 = \frac{5120 - 480\eta^2 + 27\eta^4}{5120 - 1920\eta^2} \tag{38.28}$$

$$k_1 = \frac{80 - 9\eta^2}{80 - 30\eta^2} \tag{38.29}$$

Coefficients k_1 and k_2 are presented in Table 38.1 for various values of η for angles and deflections calculated by the moment-operational method and the exact values obtained by V. Makushin and N. Malinin by means of Bessel's functions.

Table 38.1

Comparison of Coefficients k_1, k_2 and k_3

Factor η	For initial angle		For initial deflection		For maximum bending moment	
	k_1	k_1'	k_2	k_2'	k_3	k_3'
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	1.0026	1.0108	1.0028	1.0040	1.0029	1.0050
0.2	1.0107	1.0116	1.0118	1.0200	1.0115	1.0116
0.3	1.0244	1.0235	1.0260	1.0224	1.0265	1.0254
0.4	1.0447	1.0428	1.0479	1.0448	1.0490	1.0470
0.5	1.0724	1.0711	1.0779	1.0752	1.0803	1.0758
0.6	1.1094	1.1046	1.1196	1.1136	1.1234	1.1130
0.7	1.1576	1.1500	1.1699	1.1608	1.1772	1.1604
0.8	1.2211	1.2058	1.2397	1.2200	1.2598	1.2204
0.9	1.3054	1.2773	1.1322	1.2976	1.3669	1.2972
0.10	1.4200	1.3874	1.4584	1.3944	1.5175	1.3956

According to this table, the difference in the approximate and exact solutions is observed to increase together with η, i.e., with the increase of axial forces; however, even when $\eta = 1$ (which corresponds to 30% of the critical load) the divergence does not exceed 4%. Thus, two terms in the derived formulas are sufficient for most of the practical cases, although there are no essential difficulties involved in determining more exact approximations according to formulas (38.16) and (38.17). Similarly, calculations of all static and kinematic quantities, as applied to any transverse load, are reduced to algebraic computations only.

To calculate the bending moment in the second approximation we use formula (38.21) substituting into it the value of $x = l$ and for the moments of higher order, the following values

$$\frac{M_4}{4!} = P\frac{x_4}{4!} + EI\Theta_A\frac{x^2}{2} + EIv_Ax$$

$$\frac{M_3}{3!} = P\frac{x_3}{3!} + EI\Theta_A x + EIv_A$$

$$M_1 = Px$$

which results in

$$\max M = Pl\left(1 - \frac{\beta l^3}{8}\right) - EI\Theta_A\frac{\beta l^2}{2}$$

But

$$EI\Theta_A = -\frac{Pl^2}{2}k_1;\quad \beta l^3 = \frac{9}{4}\eta^2$$

we arrive at

$$\max M = Pl\left(1 - \frac{9}{32}\eta^2 + \frac{9}{16}\eta^2 k_1\right) \tag{38.30}$$

or

$$\max M = Plk_3 \tag{38.31}$$

where

$$k_3 = 1 - \frac{9}{32}\eta^2 + \frac{9}{16}\eta^2 k_1 \tag{38.32}$$

Various values of k_3 are presented in Table 38.1 for comparison with corresponding values of k_3' calculated by V. Makushin and N. Malinin by means of Bessel's functions.

Owing to the sign-varying character of the moment series and because of the rapid decrease of terms, the obtained values of static and kinematic quantities will oscillate about the exact values, if the moment series are limited to the first, second, third, etc. terms. Solutions obtained by taking the first terms only (which means assuming $\beta = 0$) will be obviously smaller than the exact ones. This follows from the fact that $\beta = 0$ corresponds to a beam without any axial loads, while the infinite series give exact solutions. Therefore, the first terms give underestimated values, while values obtained by formulas (38.18)-(38.21) are overestimated. The next approximations, if required, will be underestimated with respect to the exact solutions, although they will be much closer to the exact solution than the first approximation (underestimated) and the second approximation (overestimated).

Let us analyse the speed of convergence of one of the moment series, for example, the series represented by (38.16). It is obvious that the greater divergence from the exact solution is expected when a point load is applied to the free end. Then

$$EI\frac{dv}{dx}=EI\Theta_A\left[1-\beta\frac{x^3}{3!}+\beta^2\frac{10x^6}{6!}-\beta^3\frac{28x^9}{9!}+\ldots+\right]+$$
$$+P\left[\frac{x^2}{2}-\beta\frac{3x^5}{5!}+\beta^2\frac{18x^8}{8!}-\beta^3\frac{162x^{11}}{11!}+\ldots\right]$$

Assuming, as before, that $\beta l^3=\frac{9}{4}\eta^2$ and therefore $\beta x^3=$ $=\frac{9}{4}\eta^2\left(\frac{x}{l}\right)^3$ we arrive at

$$EI\frac{dv}{dx}=EI\Theta_A\left[1-\frac{9}{24}\eta^2\left(\frac{x}{l}\right)^3+\frac{810}{16\cdot 6!}\eta^4\left(\frac{x}{l}\right)^6-\right.$$
$$\left.-\frac{9^3\cdot 28}{4^3\cdot 9!}\eta^6\left(\frac{x}{l}\right)^9+\ldots\right]+Px^2\left[\frac{1}{2}-\eta^2\frac{3\cdot 9}{5!\cdot 4}\left(\frac{x}{l}\right)^3+\right.$$
$$\left.+\eta^4\frac{18\cdot 9^2}{8!\cdot 4^2}\left(\frac{x}{l}\right)^6-\eta^6\frac{162\cdot 9^3}{11!\cdot 4^3}\left(\frac{x}{l}\right)^9+\ldots\right] \qquad (38.33)$$

Let us take the slowest case of convergence ($x=l$) when determining the initial value of deviation when the deviation at the fixed end is equal to zero. Then the expression in the first brackets of formula (38.33) will form a series, the members of which are

$$u_1=1;\quad u_2=\frac{9}{24}\eta^2;\quad u_3=\frac{9\eta^4}{128}$$

Assuming $\eta=1$ for simplicity of calculations (Table 38.1), the following ratios are obtained for the coefficients

$$\frac{u_2}{u_1}=\frac{30}{80};\quad \frac{u_3}{u_2}=\frac{15}{80};\quad \frac{u_4}{u_3}=\frac{1}{80}\ \ldots$$

These relations transform the examined series into the following alternating, monotonically and rapidly decreasing series

$$u_1\,[1-0.375+0.0703-0.009+\ldots]$$

Similar analysis carried out for the series in the second brackets of formula (38.33), when $x=l$ and $\eta=1$, gives

$$\frac{Pl^2}{2}(1-0.1120-0.00445-0.00009+\ldots)$$

with the following ratios of the terms

$$\frac{u_2}{u_1}=\frac{9}{80}=0.112;\quad \frac{u_3}{u_2}=\frac{8}{224}=0.040$$

$$\frac{u_4}{u_3}=0.020$$

From the above we conclude that when evaluating the strength and rigidity of a beam, we may expect quite satisfactory results using only two members of the infinite moment series, i.e., from formulas (38.18)-(38.21).

If the moment symbols in (38.18)-(38.21) are replaced by their expressions according to formulas (I)-(IV) for any particular loading, for example, the loading shown in Fig. 38.1 (or a more complicated one), then the so-called "universal" formulas will be obtained.

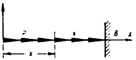

Fig. 38.3

For a beam loaded as shown in Fig. 38.2 the following expressions are given by (38.20) and (38.21) for the transverse force and the bending moment at an arbitrary section X at a distance x from the free end of the beam

$$Q=P\left(1-\frac{\beta x^3}{2}\right)-\beta xEI\Theta_A=P\left[1-\frac{9}{8}\eta^2\left(\frac{x}{l}\right)^3\right]-EI\Theta_A\frac{x^3}{l^3}\cdot\frac{9}{4}\eta^2$$

$$\begin{aligned}M&=Px-P\beta\left(x\frac{x^3}{3!}-\frac{x^4}{4!}\right)-\beta EI\Theta_A\frac{x^2}{2}=\\&=Px\left(1-\frac{\beta x^3}{8}\right)-EI\Theta_A\frac{\beta x^2}{2}=\\&=Px\left[1-\frac{9}{32}\eta^2\left(\frac{x}{l}\right)^3\right]-EI\Theta_A\frac{x^3}{l^3}\cdot\frac{8}{9}\eta^2\end{aligned}$$

where

$$\beta x^3=\beta l^3\frac{x^3}{l^3}=\frac{9}{4}\eta^2\frac{x^3}{l^3}$$

To calculate the stability of a beam subjected to axial loads (Fig. 38.3), we reject all transverse loads and take into account only the possible "kinematic" load, i.e., the initial deviation Θ_A, when

$x = l$ formula (38.16) gives

$$EI\Theta_B = EI\Theta_A\Big[1-\beta\Big(l\frac{l^2}{2}-2\frac{l^3}{3!}\Big)+ \\ +\beta^2\Big(l^2\frac{l^4}{4!}-6l\frac{l^5}{5!}+10\frac{l^6}{6!}\Big)-\beta^3\Big(l^3\frac{l^6}{6!}-12l^2\frac{l^7}{7!}+ \\ +52l\frac{l^8}{8!}-80\frac{l^9}{9!}\Big)\Big]=EI\Theta_A\Big[1-\frac{\beta l^3}{3!}+\frac{4}{6!}(\beta l^3)^2- \\ -\frac{28}{9!}(\beta l^3)^3+\ \dots\Big]=0 \qquad (38.34)$$

The right-hand part of this formula is zero only in two cases: (1) when $EI\Theta_A = 0$ or (2) when the expression in brackets is zero. The first case corresponds to small axial forces, when the load does not exceed the critical point and the axis of the beam does not change its initial rectilinear shape. Otherwise $\Theta_A \neq 0$ and the condition $\Theta_B = 0$ is satisfied if the expression in brackets is zero.

Remembering that

$$\beta l^3 = \frac{pl^3}{EI} = pl\,\frac{l^2}{EI} = \frac{Sl^2}{EI} = m \qquad (38.35)$$

where $S = pl$ is the resultant of the axial forces, from formula (38.34) we obtain the series

$$1-\frac{m}{2\cdot3}+\frac{m^2}{2\cdot3\cdot5\cdot6}-\frac{m^3}{2\cdot3\cdot5\cdot6\cdot8\cdot9}+\frac{m^4}{2\cdot3\cdot5\cdot6\cdot8\cdot9\cdot11\cdot12}\dots=0 \qquad (38.36)$$

If this series is limited to two, four, or any other even number of terms, it gives the underestimated value of the critical load. However, if the series is limited to an uneven number of terms, we obtain an overestimated value of the critical load. For example, the second approximation gives

$$1-\frac{m_2}{2\cdot3}=0;\quad m_2=6;\quad S_2=\frac{6EI}{l^2}<S=7.87\frac{EI}{l^2}$$

the third approximation gives

$$1-\frac{m_3}{2\cdot3}+\frac{m_3^2}{2\cdot3\cdot5\cdot6}=0;\quad m_3^2-5\cdot6m_3+2\cdot3\cdot5\cdot6=0$$

$$m_3=8.29;\quad S_3=\frac{8.29EI}{l^2}>S$$

the fourth approximation gives

$$1-\frac{m_4}{2\cdot3}+\frac{m_4^2}{2\cdot3\cdot5\cdot6}-\frac{m_4^3}{2\cdot3\cdot5\cdot6\cdot8\cdot9}=0$$

$$m_4=7.83;\quad S_4=7.83\,\frac{EI}{l^2}<S$$

Thus, the following sequence of approximations is obtained for the critical load

$$S_1 < S_3 < S_5 \ldots < S_{2n+1} < S_{2n+3} < \ldots < S < S_{2n+2} < \\ < S_{2n} < \ldots < S_4 < S_2 \qquad (38.37)$$

The value of the critical force obtained from the cubic equation (S_4) is about 0.5% smaller than the exact value of the critical force; this is of great practical importance. In general, when solving differential equations by the method of successive approximations we obtain a sequence, which contains alternating values (underestimating and overestimating the exact solutions), while all the solutions given by the energy method overestimate the exact solution. For example, this happens in solving a cantilever by the Ritz-Timoshenko method, if the flexural curve is assumed to be formed by loading the free end of a beam with a point load or by any other load. In addition, the calculations involved in the energy method are very large.

For convenience of use, formulas (38.13)-(38.21) are expanded for the general type of loadings shown in Fig. 38.1

$$\begin{aligned} Q = & -px\Theta_A + \left[P_i + q_i(x-c_i) + k_i \frac{(x-d_i)^2}{2} \right] - \\ & - \beta x \left[\overline{M}_i (x-a_i) + P_i \frac{(x-b_i)^2}{2} - \right. \\ & \left. - q_i \frac{(x-c_i)^3}{3!} + k_i \frac{(x-d_i)^4}{4!} \right] \end{aligned} \qquad (38.38)$$

$$\begin{aligned} M = & -\frac{px^2}{2}\Theta_A + \left[\overline{M}_i + P_i(x-b_i) + q_i \frac{(x-c_i)^2}{2} + \right. \\ & \left. + k_i \frac{(x-d_i)^3}{3!} \right] - \beta x \left[\overline{M}_i \frac{(x-a_i)^2}{2} + P_i \frac{(x-b_i)^3}{3!} + \right. \\ & \left. + q_i \frac{(x-c_i)^4}{4!} + k_i \frac{(x-d_i)^5}{5!} \right] + \beta \left[M_i \frac{(x-a_i)^3}{3!} + P_i \frac{(x-b_i)^4}{4!} + \right. \\ & \left. + q_i \frac{(x-c_i)^5}{5!} + k_i \frac{(x-d_i)^6}{6!} \right] \end{aligned} \qquad (38.39)$$

$$\begin{aligned} EI\frac{dv}{dx} = & EI\Theta_A \left[1 - \beta \frac{x^3}{6} \right] + \left[\overline{M}_i (x-a_i) + P_i \frac{(x-b_i)^2}{2} + \right. \\ & \left. + q_i \frac{(x-c_i)^3}{3!} + k_i \frac{(x-d_i)^4}{4!} \right] - \beta x \left[\overline{M}_i \frac{(x-a_i)^3}{3!} + P_i \frac{(x-b_i)^4}{4!} + \right. \\ & \left. + q_i \frac{(x-c_i)^5}{5!} + k_i \frac{(x-d_i)^6}{6!} \right] + 2\beta \left[\overline{M}_i \frac{(x-d_i)^4}{4!} + P_i \frac{(x-b_i)^5}{5!} + \right. \\ & \left. + q_i \frac{(x-c_i)^6}{6!} + k_i \frac{(x-d_i)^7}{7!} \right] \end{aligned} \qquad (38.40)$$

$$\begin{aligned}EIv_X &= EIv_A + EI\Theta_A\left[x - \frac{\beta x^4}{24}\right] + \\ &+ \left[\overline{M}_l\frac{(x-a_l)^2}{2} + P_l\frac{(x-b_l)^3}{3!} + q_l\frac{(x-c_l)^4}{4!} + k_l\frac{(x-d_l)^5}{5!}\right] - \\ &- \beta x\left[\overline{M}_l\frac{(x-a_l)^4}{4!} + P_l\frac{(x-b_l)^5}{5!} + q_l\frac{(x-c_l)^6}{6!} + k_l\frac{(x-d_l)^7}{7!}\right] + \\ &+ 3\beta\left[\overline{M}_l\frac{(x-a_l)^5}{5!} + P_l\frac{(x-b_l)^6}{6!} + q_l\frac{(x-c_l)^7}{7!} + k_l\frac{(x-d_l)^8}{8!}\right] \quad (38.41)\end{aligned}$$

These formulas determine the bending moment, rotation angles, deflection and transverse force at any arbitrary section X with abscissa x. All these quantities will be larger by a few per cent than the exact values obtained by means of Bessel's functions. Initial angles and deflections may be determined, as usual, from the boundary conditions for the fixed end (zero angle and zero deflection).

39. PRISMATIC BEAM UNDER ARBITRARY TRANSVERSE LOAD AND LINEAR AXIAL FORCES

The differential equation for the flexural curve of the particular axial load (Fig. 39.1) is

$$EI\frac{d^3v}{dx^3} + \frac{kx^2}{2}\cdot\frac{dv}{dx} = M_0$$

Substitution of $EI\gamma$ for k results in

$$EI\frac{d^3v}{dx^3} = M_0 - \frac{\gamma x^2}{2}EI\frac{dv}{dx} \tag{39.1a}$$

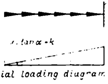

Fig. 39.1

The first approximation is derived by assuming $\gamma = 0$

$$EI\frac{d^3v}{dx^3} = M_0;\quad EI\frac{d^2v}{dx^2} = M_1 \tag{39.1b}$$

and

$$EI\frac{dv}{dx} = \frac{M_2}{2};\quad EIv = \frac{M_3}{3!} \tag{39.2}$$

Substitution of the value of $EI\frac{dv}{dx}$ obtained at the first approximation into (39.1*a*) results in

$$EI\frac{d^3v}{dx^3} = -\gamma\frac{x^2}{2}\cdot\frac{M_2}{2} \tag{39.3}$$

Integration of this equation gives

$$EI\frac{d^2v}{dx^2} = -\gamma\left(\frac{x^2}{2}\cdot\frac{M_3}{3!} - x\frac{M_4}{4!} + \frac{M_5}{5!}\right) \tag{39.4}$$

$$EI\frac{dv}{dx} = -\gamma\left(\frac{x^2}{2}\cdot\frac{M_4}{4!} - 2x\frac{M_5}{5!} + 3\frac{M_6}{6!}\right) \tag{39.5}$$

$$EIv = -\gamma\left(\frac{x^2}{2}\cdot\frac{M_5}{5!} - 3x\frac{M_6}{6!} + 6\frac{M_7}{7!}\right) \tag{39.6}$$

The next approximation is formulated by substituting $EI\,\frac{dv}{dx}$ from (39.5) into (39.1*a*) and rewriting all the terms in such a way that they are of the same type $a_{m.\,n}\frac{x^m}{m!}\cdot\frac{M_n}{n!}$, we obtain

$$EI\frac{d^3v}{dx^3} = \gamma^2\left(6\cdot\frac{x^4}{4!}\cdot\frac{M_4}{4!} - 6\frac{x^3}{3!}\cdot\frac{M_5}{5!} + 3\frac{x^2}{2}\cdot\frac{M_6}{6!}\right) \tag{39.7}$$

The following integrals are derived from this equation

$$\begin{aligned} EI\frac{d^2v}{dx^2} = \gamma^2\Big(&6\frac{x^4}{4!}\cdot\frac{M_5}{5!} - 12\frac{x^3}{3!}\cdot\frac{M_6}{6!} + \\ &+ 15\frac{x^2}{2}\cdot\frac{M_7}{7!} - 15x\frac{M_8}{8!} + 15\frac{M_9}{9!}\Big) \end{aligned} \tag{39.8}$$

$$\begin{aligned} EI\frac{dv}{dx} = \gamma^2\Big(&6\frac{x^4}{4!}\cdot\frac{M_6}{6!} - 18\frac{x_3}{3!}\cdot\frac{M_7}{7!} + \\ &+ 33\frac{x^2}{2}\cdot\frac{M_8}{8!} - 48x\frac{M_9}{9!} + 63\frac{M_{10}}{10!}\Big) \end{aligned} \tag{39.9}$$

$$\begin{aligned} EIv = \gamma^2\Big(&6\frac{x^4}{4!}\cdot\frac{M_7}{7!} - 24\frac{x^3}{3!}\cdot\frac{M_8}{8!} + \\ &+ 57\frac{x^2}{2}\cdot\frac{M_9}{9!} - 105x\frac{M_{10}}{10!} + 168\frac{M_{11}}{11!}\Big) \end{aligned} \tag{39.10}$$

Final formulas are obtained by summarizing the successive approximations

$$\begin{aligned} EI\frac{d^3v}{dx^3} = Q = M_0 - \gamma\frac{x^2}{2}\cdot\frac{M_2}{2} &+ \\ + \gamma^2\Big(6\frac{x^4}{4!}\cdot\frac{M_4}{4!} - 6\frac{x^3}{3!}\cdot\frac{M_5}{5!} &+ 3\frac{x^2}{2}\cdot\frac{M_6}{6!}\Big) - \ldots \end{aligned} \tag{39.11}$$

$$\begin{aligned} EI\frac{d^2v}{dx^2} = M = M_1 - \gamma\Big(&\frac{x^2}{2}\cdot\frac{M_3}{3!} - x\frac{M_4}{4!} + \frac{M_5}{5!}\Big) + \\ &+ \gamma^2\Big(6\frac{x^4}{4!}\cdot\frac{M_5}{5!} - 12\frac{x^3}{3!}\cdot\frac{M_6}{6!} + 15\frac{x^2}{2}\cdot\frac{M_7}{7!} - \\ &- 15x\frac{M_8}{8!} + 15\frac{M_9}{9!}\Big) - \ldots \end{aligned} \tag{39.12}$$

$$EI\frac{dv}{dx}=EI\Theta_X=\frac{M_2}{2}-\gamma\left(\frac{x^2}{2}\cdot\frac{M_4}{4!}-2x\frac{M_5}{5!}+3\,\frac{M_6}{6!}\right)+$$
$$+\gamma^2\left(6\frac{x^4}{4!}\cdot\frac{M_6}{6!}-18\,\frac{x^3}{3!}\cdot\frac{M_7}{7!}+33\,\frac{x^2}{2}\cdot\frac{M_8}{8!}-\right.$$
$$\left.-48x\frac{M_9}{9!}+63\frac{M_{10}}{10!}\right)-\dots \tag{39.13}$$

$$EIv_X=\frac{M_3}{3!}-\gamma\left(\frac{x^2}{2}\cdot\frac{M_5}{5!}-3x\frac{M_6}{6!}+6\,\frac{M_7}{7!}\right)+$$
$$+\gamma^2\left(6\,\frac{x^4}{4!}\cdot\frac{M_7}{7!}-24\,\frac{x^3}{3!}\cdot\frac{M_8}{8!}+57\,\frac{x^2}{2}\cdot\frac{M_9}{9!}-\right.$$
$$\left.-105x\frac{M_{10}}{10!}+168\frac{M_{11}}{11!}\right)-\dots \tag{39.14}$$

For practical purposes these formulas give an excessive accuracy, two terms in each formula are sufficient, as explained in the previous sections. Let us examine a case when the transverse load consists of one force applied to the free end of the beam; then we may expect the largest possible deviations from the exact solution. Taking into account the following value of the critical load for this particular type of load

$$S_{cr}=16.10\frac{EI}{l^2}$$

we arrive at

$$S_{cr}=\frac{k_{cr}l^2}{2}=16.10\;\frac{EI}{l^2}$$
$$\gamma_{cr}=\frac{k_{cr}}{EI}=\frac{32.20}{l^4};\qquad \gamma_{cr}l^4=\frac{32.20}{m} \tag{39.15}$$

where $\frac{1}{m}$ is the ratio of the applied longitudinal load and the critical load, i. e., $m=\frac{S_{cr}}{S}$.

When $x=l$ from formula (39.14) we find

$$EIv_B=0=\frac{Pl^2}{6}\left[1-\frac{32.2}{m}\cdot\frac{6}{7!}+\left(\frac{32.2}{m}\right)^2\frac{6.618}{11!}-\dots\right]+$$
$$+EI\Theta_A l\left[1-\frac{32.2}{m}\cdot\frac{1}{5!}+\left(\frac{32.2}{m}\right)^2\frac{735}{9!}-\dots\right]+EIv_A$$

For an axial load which is half the critical load, i.e., when $m=2$

$$\frac{Pl^3}{6}(1-0.0191+0.0064-\dots)+$$
$$+EI\Theta_A l(1-0.1338+0.0151-\dots)+EIv_A=0$$

The alternating series in parentheses converge rapidly, so that in all the previous cases sufficiently accurate solutions will be given by two terms of each formula (39.11)-(39.14) for the static and kinematic values of the beam strength and rigidity.

Stability of the examined beam may be also analysed, let us assume that there is no transverse load and only kinematic load need be considered (i.e., the initial value of the angle); then in accordance with (39.13) we obtain

$$EI\Theta_B = EI\Theta_A \left(1 - \gamma\left(\frac{l^2}{2}\cdot\frac{l^2}{2} - 2l\frac{l^3}{3!} + 3\,\frac{l^4}{4!}\right) + \right.$$
$$+\gamma^2\left(6\,\frac{l^4}{4!}\cdot\frac{l^4}{4!} - 18\,\frac{l^3}{3!}\cdot\frac{l^5}{5!} + 33\,\frac{l^2}{2}\cdot\frac{l^6}{6!} - \right.$$
$$\left.\left. - 48l\,\frac{l^7}{7!} + 63\,\frac{l^8}{8!}\right) - \ldots\right] = 0$$

Therefore, since $EI\Theta_A \neq 0$

$$1 - \frac{\gamma l^4}{2}\cdot\frac{1}{3\cdot 4} + \left(\frac{\gamma l^4}{2}\right)^2\frac{1}{3\cdot 4\cdot 7\cdot 8} = 0 \tag{39.16}$$

Noting that $S = \frac{kl^2}{2} = \frac{\gamma l^2 EI}{2}$ and therefore $\frac{\gamma l^4}{2} = \frac{Sl^2}{EI}$, equation (39.16) if limited only to three terms may be rearranged in the following way

$$1 - \frac{Sl^2}{EI}\cdot\frac{1}{3\cdot 4} + \left(\frac{Sl^2}{EI}\right)^2\frac{1}{3\cdot 4\cdot 7\cdot 8} = 0$$

Then

$$S = 17.41\,\frac{EI}{l^2} \tag{39.17}$$

while the exact solution is

$$S = 16.10\,\frac{EI}{l^2}$$

To obtain a more accurate value of the critical force, smaller than the exact solution and therefore preferable for practical purposes, the following cubic equations should be solved

$$1 - \frac{Sl^2}{EI}\cdot\frac{1}{3\cdot 4} + \left(\frac{Sl^2}{EI}\right)^2\frac{1}{3\cdot 4\cdot 7\cdot 8} - \left(\frac{Sl^2}{EI}\right)^3\frac{1}{3\cdot 4\cdot 7\cdot 8\cdot 11\cdot 12} = 0$$

The solution is

$$\frac{Sl^2}{EI} = 16.03$$

therefore

$$S = 16.03\,\frac{EI}{l^2} \tag{39.18}$$

which is 0.4% lower than the exact solution.

Stability problems are studied in detail in Sec. 41. Let us now design a uniform beam subjected to compression and flexure, which may be represented by a charging screw of a macaroni extruder.

Longitudinal and Transverse Deflection of a Screw

Screw conveyors are widely used in various branches of industry, such as: macaroni industry (extruding macaroni products), rubber industry (forming rubber tubes), pencil industry (making pencil leads), and in the peat, ceramic, soap-making industries, etc.

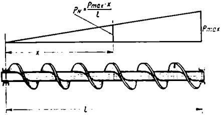

Fig. 39.2

Screw conveyors set up the working pressure by rotation of the screw in a chamber, which terminates in a die pressing head. At pressures higher than 100 kgf/cm^2 (as in extruding macaroni products) any further increase in the productivity of the extruder is limited by the strength of the charging screw.

When designing the screw, all the loads applied must be analysed and their influence upon the strength of the screw evaluated.

There are no general formulas for all the possible pressing materials and designs of the charging screw, although the methods of calculation have much in common.

As an example, let us study the design of a charging screw of a a macaroni press.

Examine a screw of a uniform screw pitch with bearings at the ends to determine the loads acting upon the screw. Such a screw is shown schematically in Fig. 39.2. The productivity of the press and the maximum pressure are assumed to be known. Pressure is assumed to vary uniformly along the screw from zero to the working pressure, which is confirmed by experimental data.

The origin of coordinates is placed at the centre of gravity of the left-hand butt-end of the screw and the axis is directed along its axle, thus forming a right-hand coordinate system. The coordinate axes are assumed to rotate together with the screw.

The normal pressure p_N at any section is determined by the following formula

$$p_N = \frac{p_{\max} x}{l} \tag{39.19}$$

where $p_{\max}$ = maximum pressure and
l = length of the screw

The intensity of the continuous axial load is determined from the following.

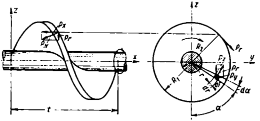

Fig. 39.3

The normal pressure p_N may be expanded into components (Fig. 39.3) as for a screw

$$p_x = p_N \cos\beta; \quad p_r = p_N \sin\beta$$

where β = mean pitch angle
p_x = axial pressure
p_r = tangent pressure

Take an elementary area dF on the surface of the screw convolution

$$dF = r d\alpha\, dr \frac{1}{\cos\beta}$$

The axial force within angle $d\alpha$ is given by

$$dN_x = \int p_x\, dF = \int_{R_2}^{R_1} p_N r\, d\alpha\, dr$$

and, as the pressure may be assumed constant along the radius,

$$dN_x = p_N \frac{R_1^2 - R_2^2}{2} d\alpha$$

where R_2 = internal radius of the screw
R_1 = external radius of the screw

Angle α has the following relationship with the abscissa x

$$\alpha = \frac{2\pi}{t}\, x$$

where t = screw pitch.

Thus, the intensity of the continuous axial load is given by

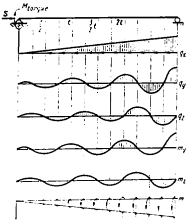

Fig. 39.4

$$q_x = \frac{dN_x}{dx} = p_x \frac{R_1^2 - R_2^2}{2} \cdot \frac{2\pi}{t} \tag{39.20}$$

The axial pressure p_x in addition to the axial compressive stress creates bending moments about y and z axes; these distributed moments are determined in a similar manner and are correspondingly equal to

$$m_z = \frac{dM_z}{dx} = p_x \frac{R_1^3 - R_2^3}{3} \cdot \frac{2\pi}{t} \sin \frac{2\pi}{t}\, x \tag{39.21}$$

$$m_y = \frac{dM_y}{dx} = p_x \frac{R_1^3 - R_2^3}{3} \cdot \frac{2\pi}{t} \cos \frac{2\pi}{t}\, x \tag{39.22}$$

The tangent pressure p_r creates a distributed torque of intensity m_{torque} and transverse loads q_y and q_z in planes of y and z axes.

$$m_{torque} = \frac{dM_{torque}}{dx} = p_x \tan\beta \frac{R_1^3 - R_2^3}{3} \cdot \frac{2\pi}{t} \tag{39.23}$$

$$q_y = p_x \tan\beta \frac{R_1^2 - R_2^2}{2} \cdot \frac{2\pi}{t} \cos \frac{2\pi}{t}\, x \tag{39.24}$$

$$q_z = p_x \tan\beta \frac{R_1^2 - R_2^2}{2} \cdot \frac{2\pi}{t} \sin \frac{2\pi}{t}\, x \tag{39.25}$$

Besides these loads, the screw is subjected to a point load S, which is the axial reaction of the left collar thrust bearing and a torque at the left support, which is transmitted to the screw by a gear.

As a result, the screw is under the influence of compression, torque and longitudinal and transverse flexure, as shown in Fig. 39.4.

Elastic Equilibrium of a Screw Under Axial and Transverse Flexure

To simplify the investigation, the transverse loads are assumed to follow constant amplitude sine and cosine law having equal average values q_y, q_z, m_y, m_z.

Then the expressions for the transverse loads in compliance with formulas (39.21), (39.22), (39.24), (39.25) are

$$q_y = \frac{q_{\max}}{2} \tan\beta \, \frac{R_1^2 - R_2^2}{2} \alpha \cos \alpha x \tag{39.26}$$

$$q_z = \frac{q_{\max}}{2} \tan\beta \, \frac{R_1^2 - R_2^2}{2} \alpha \sin \alpha x \tag{39.27}$$

$$m_y = \frac{q_{\max}}{2} \cdot \frac{R_1^3 - R_2^3}{3} \alpha \cos \alpha x \tag{39.28}$$

$$m_z = \frac{q_{\max}}{2} \cdot \frac{R_1^3 - R_2^3}{3} \alpha \sin \alpha x \tag{39.29}$$

where

$$\alpha = \frac{2\pi}{l}; \quad p_x = \frac{q_{\max}}{2}$$

Since the transverse load acts in two mutually perpendicular planes, combined compression and flexure must be studied for these two planes separately and then summarized.

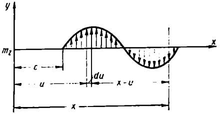

Fig. 39.5

The axial load consists of a point load (reaction of the left support) and of a uniformly increasing distributed axial load. The transverse flexure about axis z is caused by a distributed transverse load of intensity q_y and of a distributed bending moment of intensity m_z. Let us examine the combined compression and flexure in this plane.

First of all, a moment series of the n-th order for the load q_y will be composed for the general case, where the load begins at distance c from the origin, as shown in Fig. 39.5. The general formula for the n-th moment in this case is given by

$$M_n = \int_{u=c}^{u=x} q_y \, du \, (x-u)^n = \int_{u=c}^{u=x} A \, (x-u)^n \cos \alpha x \, du$$

where

$$A = \frac{q_{\max}}{2} \tan\beta \, \frac{R_1^2 - R_2^2}{2} \alpha \tag{39.30}$$

Integrating by parts and dividing both sides of the equation by $n!$ leads to

$$\frac{M_n}{n!} = \frac{A}{a} \cdot \frac{(x-u)^n}{n!} \sin a(u-c) \Bigg|_{u=c}^{u=x} - \frac{A}{a^2} \cdot \frac{(x-u)^{n-1}}{(n-1)!} \cos a(u-c) \Bigg|_{u=c}^{u=x} -$$

$$- \frac{A}{a^3} \cdot \frac{(x-u)^{n-2}}{(n-2)!} \sin a(u-c) \Bigg|_{u=c}^{u=x} + \frac{A}{a^4} \cdot \frac{(x-u)^{n-3}}{(n-3)!} \times$$

$$\times \cos a(u-c) \Bigg|_{u=c}^{u=x} + \frac{A}{a^4 (n-4)!} \Bigg|_{u=c}^{u=x} (x-a)^{n-4} \cos a(u-c)\, du \quad (39.31)$$

The general expression for M_n from the distributed bending moment m_z is given by

$$M_n = \int_{u=c}^{u=x} m_z (x-u)^{n-1} du = \int_{u=c}^{u=x} B [\sin a(u-c)\, (x-u)^{n-1} du] \quad (39.32)$$

where

$$B = \frac{q_{\max}}{2} \cdot \frac{R_1^3 - R_2^3}{3}\, a$$

If this expression is integrated as the previous formula, then the distributed bending moment results in a formula for the $(n+1)$-th moment. That is why the power of the term containing x in (39.32) must be reduced by a unity to obtain an expression for the n-th moment

$$\frac{M_n}{n!} = - \frac{B}{a} \cdot \frac{(x-u)^{n-1}}{(n-1)!} \cos a(u-c) \Bigg|_{u=c}^{u=x} -$$

$$- \frac{B}{a^2} \cdot \frac{(x-u)^{n-2}}{(n-2)!} \sin a(u-c) \Bigg|_{u=c}^{u=x} + \frac{B}{a^3} \cdot \frac{(x-u)^{n-3}}{(n-3)!} \times$$

$$\times \cos \alpha(u-c) \Bigg|_{u=c}^{u=x} + \frac{B}{a^4} \cdot \frac{(x-u)^{n-4}}{(n-4)!} \sin a(u-c) \Bigg|_{u=c}^{u=x} +$$

$$+ \frac{B}{a^4 (n-5)!} \int_{u=c}^{u=x} (x-u)^{n-5} \sin a(u-c)\, du \quad (39.33)$$

According to (39.31) and (39.33), the moment series from cads q_y and m_z after transformation become

$$\frac{M_1}{1} = \left[\frac{B}{a} + \frac{A}{a^2} \right] [1 - \cos a(x-c)] \quad (39.34)$$

$$\frac{M_2}{2!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[(x-c)-\frac{1}{a}\sin a\,(x-c)\right] \tag{39.35}$$

$$\frac{M_3}{3!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{(x-c)^2}{2!}-\frac{1}{a^2}+\frac{1}{a^2}\cos a\,(x-c)\right] \tag{39.36}$$

$$\frac{M_4}{4!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{(x-c)^3}{3!}-\frac{1}{a^2}(x-c)+\frac{1}{a^3}\sin a\,(x-c)\right] \tag{39.37}$$

$$\frac{M_5}{5!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{(x-c)^4}{4!}-\frac{1}{a^2}\cdot\frac{(x-c)^2}{2!}-\frac{1}{a^4}\cos a\,(x-c)+\frac{1}{a^4}\right] \tag{39.38}$$

$$\frac{M_6}{6!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{(x-c)^5}{5!}-\frac{1}{a^2}\cdot\frac{(x-c)^3}{3!}+\right.$$
$$\left.+\frac{1}{a^4}(x-c)-\frac{1}{a^5}\sin a\,(x-c)\right] \tag{39.39}$$

$$\frac{M_7}{7!}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{(x-c)^6}{6!}-\frac{1}{a^2}\cdot\frac{(x-c)^4}{4!}+\right.$$
$$\left.+\frac{1}{a^4}\cdot\frac{(x-c)^2}{2!}+\frac{1}{a^6}\cos a\,(x-c)-\frac{1}{a^6}\right] \tag{39.40}$$

The following relationship is observed in combined compression and flexure

$$EI\frac{d^2v}{dx^2}=M+M_1 \tag{39.41}$$

where v = deflection
EI = rigidity of the cross section
M = bending moment at section X due to axial loads
M_1 = bending moment at section X due to transverse loads

Here M_1 designates the bending moment from all the transverse loads; it does not depend on the deflection v. The bending moment from the axial loads depends on the deflection, which is an unknown quantity.

Differentiation of (39.41) gives

$$EI\frac{d^3v}{dx^3}=Q+Q_1 \tag{39.42}$$

where $Q=\frac{dM}{dx}$ = transverse force due to axial loads

$Q_1=\frac{dM_1}{dx}$ = transverse force due to transverse loads

According to Fig. 39.6,

$$Q=-S\frac{dv}{dx}+k\frac{x^2}{2!}\cdot\frac{dv}{dx} \tag{39.43}$$

where k is the slope of increasing axial load.

Substitution of the expression for Q from (39.43) in (39.42) gives

$$EI\frac{d^3v}{dx^3}=-S\frac{dv}{dx}+k\frac{x^2}{2!}\cdot\frac{dv}{dx}+Q_1$$

or

$$EI\frac{d^3v}{dx^3}+S\frac{dv}{dx}-k\frac{x^2}{2!}\cdot\frac{dv}{dx}=Q_1 \qquad (39.44)$$

This differential equation may be solved by successive approximations anb an appropriate number of approximations must be made to obtain the required accuracy.

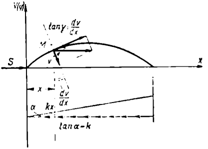

Fig. 39.6

In the first approximation the axial loads are disregarded; then equation (39.44) becomes

$$EI\frac{d^3v}{dx^3}=Q_1 \qquad (39.45)$$

Three successive integrations of (39.45) give the following expressions for the first approximation

(a) for the bending moment

$$EI\frac{d^2v}{dx^2}=M_1 \qquad (39.46)$$

(b) for the rotation angle

$$EI\frac{dv}{dx}=\frac{M_2}{2!} \qquad (39.47)$$

(c) for the deflection

$$EIv=\frac{M_3}{3!} \qquad (39.48)$$

The second approximation is determined by disregarding the transverse loads; this transforms equation (39.44) into

$$EI\frac{d^3v}{dx^3}=-S\frac{dv}{dx}+k\frac{x^2}{2!}\cdot\frac{dv}{dx} \qquad (39.49)$$

For simplicity of calculations, the following designations are introduced

$$S = \gamma^2 EI \tag{39.50}$$

$$k = \psi^2 EI \tag{39.51}$$

and substitutions into (39.49) give

$$EI \frac{d^3 v}{dx^3} = -\gamma^2 EI \frac{dv}{dx} + \psi^2 EI \frac{dv}{dx} \cdot \frac{x^2}{2!} \tag{39.52}$$

Substitution of the first approximation of $EI \frac{dv}{dx}$ from (39.47) into (39.52) gives the following equation

$$EI \frac{d^3 v}{dx^3} = -\gamma^2 \frac{M_2}{2!} + \psi^2 \frac{x^2}{2!} \frac{M_2}{2!} \tag{39.53}$$

This equation is integrated according to the rules of bimoment integration. In this particular case $\frac{x^2}{2!} \frac{M_n}{n!}$ is the bimoment to integrate; for the triple integration we use the following formulas

$$\int \frac{x^2}{2!} \cdot \frac{M_n}{n!} dx = \frac{x^2}{2!} \cdot \frac{M_{n+1}}{(n+1)!} - x \frac{M_{n+2}}{(n+2)!} + \frac{M_{n+3}}{(n+3)!} \tag{39.54}$$

$$\int\int \frac{x^2}{2!} \cdot \frac{M_n}{n!} dx^2 = \frac{x^2}{2!} \cdot \frac{M_{n+2}}{(n+2)!} - 2x \frac{M_{n+3}}{(n+3)!} + 3 \frac{M_{n+4}}{(n+4)!} \tag{39.55}$$

$$\int\int\int \frac{x^2}{2!} \cdot \frac{M_n}{n!} dx^3 = \frac{x^2}{2!} \cdot \frac{M_{n+3}}{(n+3)!} - 3x \frac{M_{n+4}}{(n+4)!} + 6 \frac{M_{n+5}}{(n+5)!} \tag{39.56}$$

As a result of integration of expression (39.53) in the second approximation we obtain

(a) for the bending moment

$$EI \frac{d^2 v}{dx^2} = -\gamma^2 \frac{M_3}{3!} + \psi^2 \frac{x^2}{2!} \cdot \frac{M_3}{3!} - \psi^2 x \frac{M_4}{4!} + \psi^2 \frac{M_5}{5!} \tag{39.57}$$

(b) for the rotation angle

$$EI \frac{dv}{dx} = -\gamma^2 \frac{M_4}{4!} + \psi^2 \frac{x^2}{2!} \cdot \frac{M_4}{4!} - 2x\psi^2 \frac{M_5}{5!} + 3\psi^2 \frac{M_6}{6!} \tag{39.58}$$

(c) for the deflection

$$EIv = -\gamma^2 \frac{M_5}{5!} + \psi^3 \frac{x^2}{2!} \cdot \frac{M_5}{5!} - \psi^2 3x \frac{M_6}{6!} + 6\psi^3 \frac{M_7}{7!} \tag{39.59}$$

Suppose two approximations are sufficient. More accurate solutions may be obtained by taking the third and fourth approximations, i.e., by substituting the expression for the angle from the second approximation (39.58) into the initial equation (39.52) and by integrating the obtained expression three times, as in the second approximation. Any required accuracy may be obtained by this method.

Final expressions for the second approximations are obtained by summarizing the right-hand parts of the corresponding equations from the first and second approximations:

(a) for the bending moment

$$EI\frac{d^2v}{dx^2} = M_1 - \gamma^2\frac{M_3}{3!} + \psi^3\frac{x^2}{2!}\cdot\frac{M_3}{3!} - \psi^2 x\frac{M_4}{4!} + \psi^2\frac{M_5}{5!} \qquad (39.60)$$

(b) for the rotation angle

$$EI\frac{dv}{dx} = \frac{M_2}{2!} - \gamma^2\frac{M_4}{4!} + \psi^2\frac{x^2}{2!}\cdot\frac{M_4}{4!} - \psi^2 2x\frac{M_5}{5!} + 3\psi^2\frac{M_6}{6!} \qquad (39.61)$$

(c) for the deflection

$$EIv = \frac{M_3}{3!} - \gamma^2\frac{M_5}{5!} + \psi^2\frac{x^2}{2!}\cdot\frac{M_5}{5!} - \psi^2 3x\frac{M_6}{6!} + 6\psi^2\frac{M_7}{7!} \qquad (39.62)$$

These formulas give approximate solutions for uniform beams by the moment-operational method for a composite load, which consists of an arbitrary transverse load, compressing point load S and a uniformly distributed varying axial load.

The general expression for $\frac{M_n}{n!}$ in these formulas may be expanded for any particular load applied to the worm, including all static factors and initial kinematic factors (initial angle and deflection), according to the following formula

$$\frac{M_n}{n!} = EIv_0\frac{x^{n-3}}{(n-3)!} + EI\Theta_0\frac{x^{n-2}}{(n-2)!} + \sum\overline{M}\frac{(x-x_M)^{n-1}}{(n-1)!} +$$
$$+ \sum P\frac{(x-x_p)^n}{n!} + \sum q\frac{(x-x_q)^{n+1}}{(n+1)!} + \sum k\frac{(x-x_k)^{n+2}}{(n+2)!} + \dots \qquad (39.63)$$

where v_0, Θ_0 = deflection and angle at the origin

$\overline{M}$, P, q, k = static factors of the transverse load

For the case under examination, i.e., the given worm loading and the location of supports, quantities v_0, $\overline{M}$, P, q and k are zero, but there is a transverse distributed load following a cosine law and a distributed bending moment following a sine law.

Equation (39.60) may be used for evaluating the strength of the worm, with the expressions from (39.36), (39.37), and (39.38) substituted for corresponding moment symbols $\frac{M_n}{n!}$ at $c = 0$ as the load is assumed acting upon the entire worm

$$EI\frac{d^2v}{dx^2} = \left[\frac{B}{a} + \frac{A}{a^2}\right][1 \cos ax] - \gamma^2\left[\frac{B}{a} + \frac{A}{a^2}\right]\times$$
$$\times\left[\frac{x^2}{2!} + \frac{1}{a^2}\cos ax - \frac{1}{a^2} + EI\Theta_0 x\right] + \dots \qquad (39.64)$$

After collecting terms we obtain

$$EI\frac{d^2v}{dx^2}=\left[\frac{B}{a}+\frac{A}{a^2}\right][1-\cos ax]-\gamma^2 EI\Theta_0 x+$$
$$+\psi^2\frac{x^3}{3!}EI\Theta_0-\gamma^2\left[\frac{B}{a}+\frac{A}{a^2}\right]\left[\frac{x^2}{2!}-\frac{1}{a^2}\cos ax-\frac{1}{a^4}\right]+\psi^2\left[\frac{B}{a}+\right.$$
$$\left.+\frac{A}{a^2}\right]\left[3\frac{x^4}{4!}+\frac{x^2}{2!}\cdot\frac{1}{a^2}\cos ax-\frac{x}{a^3}\sin ax+\frac{1}{a^4}(1-\cos ax)\right] \quad (39.65)$$

Finally, new designations are introduced by

$$f_1(x)=1-\cos ax$$
$$f_2(x)=\frac{x^2}{2}+\frac{1}{a^2}\cos ax-\frac{1}{a^2}$$
$$f_3(x)=3\frac{x^4}{4!}+\frac{x^2}{2}\cdot\frac{1}{a^4}\cos ax-\frac{x}{a^3}\sin ax+\frac{1}{q^4}(1-\cos ax)$$

which are substituted into (39.65) to give the following expression

$$EI\frac{d^2v}{dx^2}=\left[\frac{B}{a}+\frac{A}{a^2}\right]\cdot f_1(x)-\gamma^2\left[\frac{B}{a}-\frac{A}{a^2}\right]f_2(x)+$$
$$+\psi^2\left[\frac{B}{a}+\frac{A}{a^2}\right]f_3(x)-\gamma^2 xEI\Theta_0+\psi^2\frac{x^3}{3!}EI\Theta_0 \quad (39.66)$$

Tables may be compiled for $f_1(x)$, $f_2(x)$ and $f_3(x)$ for the given dimensions of the charging worm, and quantities A, B, γ^2 and ψ^2 may be determined if the pressure $p_{\max}$ is known.

40. PRISMATIC BEAM UNDER ARBITRARY TRANSVERSE LOAD AND AXIAL FORCES DISTRIBUTED ALONG THE BEAM ACCORDING TO POLYNOMIAL LAW

The differential equation of the flexural curve of a beam under the influence of a load shown in Fig. 40.1 is represented by

$$EI\frac{d^3v}{dx^3}+\left(\beta_0+\beta_1 x+\beta_2\frac{x^2}{2}+\beta_3\frac{x^3}{3!}+\dots\right)EI\frac{dv}{dx}=M_0 \quad (40.1)$$

where $N_x=(\beta_0+\beta_1 x+\beta_2\frac{x^2}{2}+\beta_3\frac{x^3}{3!}+\dots)\ EI$ is the axial load at section X with distance x from the free end of the beam. According to the conditions set in this problem, there are distributed axial loads acting upon the beam in addition to the point load at its end. The axial load distribution may be expressed or approximated by a polynomial of a certain order.

Assuming for the first approximation that the axial load is absent, we obtain the solution from formula (40.1), then substituting $\frac{M_z}{2}$ for $EI\frac{dx}{dv}$ and integrating once, twice and three times, we obtain the values of the second approximation for $\frac{d^2v}{dx^2}$, $\frac{dv}{dx}$ and v. The sum of solutions obtained in the first and second approximations gives formulas (40.2)-(40.5) which may be applied when the beam

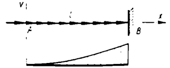

Fig. 40.1

is loaded by a compound transverse load and by axial forces distributed along the beam according to a polynomial law.

$$Q = EI\,\frac{d^3v}{dx^3} = M_0 - \left(\beta_0 + \beta_1 x + \beta_2\,\frac{x^2}{2} + \beta_3\,\frac{x^3}{3!} + \ldots\right)\frac{M_2}{2} \quad (40.2)$$

$$M = EI\,\frac{d^2v}{dx^2} = M_1 - \left[\beta_0\,\frac{M_3}{3!} + \beta_1\left(x\,\frac{M_3}{3!} - \frac{M_4}{4!}\right) + \right.$$
$$\left. + \beta_2\left(\frac{x^2}{2!}\cdot\frac{M_3}{3!} - x\,\frac{M_4}{4!} + \frac{M_5}{5!}\right) + \ldots\right] \quad (40.3)$$

$$EI\Theta_X = EI\,\frac{dv}{dx} = \frac{M_2}{2} - \left[\beta_0\,\frac{M_4}{4!} + \beta_1\left(x\,\frac{M_4}{4!} - 2\,\frac{M_5}{5!}\right) + \right.$$
$$\left. + \beta_2\left(\frac{x^2}{2}\cdot\frac{M_4}{4!} - 2x\,\frac{M_5}{5!} + 3\,\frac{M_6}{6!}\right) + \ldots\right] \quad (40.4)$$

$$EIv_X = \frac{M_3}{3!} - \left[\beta_0\,\frac{M_5}{5!} + \beta_1\left(x\,\frac{M_5}{5!} - 3\,\frac{M_6}{6!}\right) + \right.$$
$$\left. + \beta_2\left(\frac{x^2}{2}\cdot\frac{M_5}{5!} - 3x\,\frac{M_6}{6!} + 6\,\frac{M_7}{7!}\right) + \ldots\right] \quad (40.5)$$

These formulas should be expanded for the given load according to the primary formulas (I)-(III).

Solutions obtained from formulas (40.2)-(40.5) are slightly larger than the exact solutions given by the infinite moment series. All the particular solutions obtained in the previous sections show that the error caused by limiting the series to two terms is very small.

Formulas (40.2)-(40.5) are used in strength and rigidity design of bars. Formula (40.4) should be used for stability computations; it results in a quadratic equation for the critical load (overestimated), if the boundary condition is to be satisfied at the right support, i.e., the angle is zero when $x = l$ (point X is chosen at point B). However, the ratio of coefficients β_0, β_1, β_2, ... must be kept constant so that they may be expressed in terms of one parameter, which will finally give the value of the critical load. In order to obtain a slightly underestimated value of the critical load, the fourth approximation of (40.4) must be used for writing a cubic equation, as it was done in the case of formula (39.18).

Chapter VIII

Application of Moment-Operational Method to Certain Complex Problems

41. STABILITY OF BARS UNDER AXIAL COMPRESSION

For stability design of bars special subsidiary formulas must be derived which are more suitable for this purpose than the formulas applied in the previous chapter for strength and rigidity design of beams under combined action of compression and bending.

Let us consider a function of the following type

$$y = \frac{x^{n+1}}{(n+1)!} \cdot \frac{M_2}{2} \tag{41.1}$$

and suppose that successive integrals of this function are required. Repeated integration by parts results in

$$\begin{gathered}\int \frac{x^{n+1}}{(n+1)!} \cdot \frac{M_2}{2}\, dx = \int \frac{M_2}{2!}\, d\,\frac{x^{n+2}}{(n+2)!} = \frac{M_2}{2} \cdot \frac{x^{n+2}}{(n+2)!} - \\ - \int \frac{x^{n+2}}{(n+2)!} M_1\, dx = \frac{M_2}{2} \cdot \frac{x^{n+2}}{(n+2)!} - \int M_1 d \frac{x^{n+3}}{(n+3)!} = \\ = \frac{M_2}{2} \cdot \frac{x^{n+2}}{(n+2)!} - M_1 \frac{x^{n+3}}{(n+3)!} + M_0 \frac{x^{n+4}}{(n+4)!} - M_{-1} \frac{x^{n+5}}{(n+5)!} + \ldots\end{gathered} \tag{41.2}$$

No transverse load is assumed to be applied to the bar, as shown in Fig. 41.1; there is only a kinematic load at the free end A of the bar (initial slope and deflection), and an axial load following a certain law is applied to the bar. Under these particular conditions all the moments, except M_2, will vanish and the second moment will have a constant value equal to the initial slope

$$\frac{M^2}{2} = EI\Theta_A \tag{41.3}$$

Hence

$$\int \frac{x^{n+1}}{(n+1)!} \cdot \frac{M_2}{2}\, dx = \frac{M_2}{2} \cdot \frac{x^{n+2}}{(n+2)!} \tag{41.4}$$

The successive integrals can be accordingly expressed as

$$\int\int \frac{x^{n+1}}{(n+1)!} \cdot \frac{M_2}{2}\, dx^2 = \frac{M_2}{2} \cdot \frac{x^{n+3}}{(n+3)!} \tag{41.5}$$

and

$$\int\int\int \frac{x^{n+1}}{(n+1)!} \cdot \frac{M_2}{2}\, dx^3 = \frac{M_2}{2} \cdot \frac{x^{n+4}}{(n+4)!} \tag{41.6}$$

Formulas (41.2)-(41.6) cannot be applied to transverse static loads, because functions M_1, M_0, M_{-1}, . . ., will be discontinuous as in the case of concentrated static loads, such as: a force couple, a point load, a distributed load. These functions will have two values at their application points; thus, formulas (41.2)-(41.6) may be used only when the distributed loads are applied to the entire length of the bar and initial kinematic parameters are allowed for. In these cases formulas (41.2)-(41.6) will be even more effective than those presented in Sec. 19.

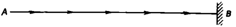

Fig. 41.1

Formulas (41.2)-(41.6) will be applied to determine critical loads for a uniform rod fixed at one end (Fig. 41.1) and subjected to a parabolic axial load. Suppose the axial force at section X at a distance x from the free end is given by

$$N_X = S\frac{x^n}{l^n} \tag{41.7}$$

where S is the resultant of all axial loads applied to the bar within the span.

By denoting

$$S = \alpha EI \tag{41.8}$$

we arrive at a differential equation for the flexural curve, as in examining beams under combined action of compression and bending

$$EI\frac{d^3v}{dx^3} + \alpha\frac{x^n}{l^n}EI\frac{dv}{dx} = M_0 \tag{41.9}$$

But $M_0 = 0$ in the absence of transverse static load; therefore

$$EI\frac{d^3v}{dx^3} = -\alpha\frac{x^n}{l^n}EI\frac{dv}{dx} \tag{41.10}$$

Taking for the first approximation, $EI\frac{dv}{dx}=\frac{M_2}{2}$, we obtain

$$EI\frac{d^3v}{dx^3}=-\alpha\frac{x^n}{l^n}\cdot\frac{M_2}{2} \tag{41.11}$$

and the two successive integrals will be given by

$$EI\frac{d^2v}{dx^2}=-\alpha\frac{x^{n+1}}{(n+1)\,l^n}\cdot\frac{M_2}{2} \tag{41.12}$$

and

$$EI\frac{dv}{dx}=-\alpha\frac{x^{n+2}}{(n+1)(n+2)\,l^n}\cdot\frac{M_2}{2} \tag{41.13}$$

Substitution of this expression in the right-hand side of (41.10) will result in

$$EI\frac{d^3v}{dx^3}=\alpha^2\frac{x^{2n+2}}{l^{2n}(n+1)(n+2)}\cdot\frac{M_2}{2} \tag{41.14}$$

and the second integral of this expression will give for the second approximation

$$EI\frac{dv}{dx}=\alpha^2\frac{x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)\,l^{2n}}\cdot\frac{M_2}{2}\text{, etc.} \tag{41.15}$$

As a result, the following infinite series is obtained

$$\begin{aligned}EI\Theta=\frac{M_2}{2}\Big[1&-\frac{\alpha}{l^n}\cdot\frac{x^{n+2}}{(n+1)(n+2)}+\\&+\frac{\alpha^2}{l^{2n}}\cdot\frac{x^{2n+4}}{(n+1)(n+2)(2n+3)(2n+4)}-\\&-\frac{\alpha^3}{l^{3n}}\frac{x^{3n+6}}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)}+\dots\Big]\end{aligned} \tag{41.16}$$

Since the slope is zero at the fixed end, the right-hand side of (41.16) is zero when $x=l$, which means that either $\Theta_B=0$ or the expression in the brackets is zero. In our case only the second version is possible, and so

$$\begin{aligned}1&-\frac{\alpha l^2}{(n+1)(n+2)}+\frac{(\alpha l^2)^2}{(n+1)(n+2)(2n+3)(2n+4)}-\\&-\frac{(\alpha l^2)^3}{(n+1)(n+2)(2n+3)(2n+4)(3n+5)(3n+6)}+\dots=0\end{aligned} \tag{41.17}$$

By taking only three terms of (41.17) we obtain a quadratic equation for the critical load

$$1-\frac{\alpha l^2}{(n+1)(n+2)}+\frac{(\alpha l^2)^2}{(n+1)(n+2)(2n+3)(2n+4)}=0 \tag{41.18}$$

By assuming $n=0,1,2,3$, etc., we derive from (41.18) a set of quadratic equations corresponding to various types of distributed axial load. According to (41.8)

$$S=\alpha EI=\alpha l^2 \frac{EI}{l^2}=k \frac{EI}{l^2} \tag{41.19}$$

where $k=\alpha l^2$.

Therefore,

(1) $k_0^2-3\cdot 4k_0+1\cdot 2\cdot 3\cdot 4=0$ when $n=0$ (41.20)

(2) $k_1^2-5\cdot 6k_1+2\cdot 3\cdot 5\cdot 6=0$ when $n=1$ (41.21)

(3) $k_2^2-7\cdot 8k_2+3\cdot 4\cdot 7\cdot 8=0$ when $n=2$ (41.22)

(4) $k_3^2-9\cdot 10k_3+4\cdot 5\cdot 9\cdot 10=0$ when $n=3$ (41.23)

Solving these quadratic equations and taking the lowest values of the critical loads, we obtain the following cases.

(1) When $n=0$, $k_0=2.54$ and

$$S=2.54 \frac{EI}{l^2} \tag{41.24}$$

However, the case when $n=0$ means that the bar is subjected to a constant point force and the critical load is known to be equal to

$$S=\frac{\pi^2 EI}{4l^2}=2.47 \frac{EI}{l^2} \tag{41.25}$$

This should be compared with the approximate solution given by the Ritz-Timoshenko method. If the flexural curve is assumed to follow the shape of a cantilever loaded by a transverse force at the free end, then the critical force is

$$S=2.50 \frac{EI}{l^2} \tag{41.26}$$

Thus, the moment-operational method gives a solution which deviates from the exact solution by 3% and from the Ritz-Timoshenko solution, by 1.5%, as follows from comparison of formulas (41.24)-(41.26).

(2) When $n=1$, $k_1=8.29$ and

$$S=8.29 \frac{EI}{l^2} \tag{41.27}$$

These conditions refer to a uniformly distributed axial load, and the approximate solutions given by S. Timoshenko and J. Lessels are the following

$$S_1=\frac{\pi^2 EI}{1.19l^2}=8.30 \frac{EI}{l^2} \tag{41.28}$$

$$S_2=\frac{\pi EI}{1.25l^2}=7.90 \frac{EI}{l^2} \tag{41.29}$$

The first approximate solution (41.28) practically does not differ from the solution given by (41.27), while the second solution (41.29) deviates from it by 4.7%; as for the amount of calculations involved, the moment-operational method appears to be preferable.

(3) When $n = 2$, $k_3 = 17.41$ and

$$S = 17.41 \frac{EI}{l^2} \tag{41.30}$$

This case corresponds to a triangular diagram of the axial load.

(4) When $n = 3$, $k_4 = 30.0$ and

$$S = 30 \frac{EI}{l^2} \tag{41.31}$$

This formula determines the critical load in the case of axial forces distributed along the beam according to the square parabola law.

The series given by (41.17) is obviously converging, because it is an alternating, monotonously decreasing series and the ratio of two successive items approaches zero (when $i \to \infty$)

$$p = \frac{u_i}{u_{i-1}} = \frac{\alpha l^2}{(ni + 2i - 1)(ni + 2i)} \to 0; \quad u_i \to 0 \tag{41.32}$$

If four terms are taken in formula (41.17) instead of three, then a more exact solution is obtained whose value is slightly below the exact solution, while that of the previous solution was slightly larger.

For example, the second loading diagram examined previously (the uniform loading) leads to the following cubic equation for the more exact value of the critical load

$$1 - \frac{k}{2 \cdot 3} + \frac{k^2}{2 \cdot 3 \cdot 5 \cdot 6} - \frac{k^3}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} = 0 \tag{41.33}$$

or

$$k^3 - 8 \cdot 9k^2 + 5 \cdot 6 \cdot 8 \cdot 9k - 2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 = 0$$

$$k = 7.83 \text{ and } S = 7.83 \frac{EI}{l^2} \tag{41.34}$$

while the solution given by S. Timoshenko in the second approximation is

$$S = 7.90 \frac{EI}{l^2}$$

that is only 1% larger.

By taking two, three and four terms in (41.33) we obtain the following values of the coefficients: 6.00, 8.29, 7.83, respectively, thus approaching the exact value 7.87.

The successive solutions systematically approach the exact solution from the left (S_1, S_3) and from the right (S_2, S_4), as shown in Fig. 41.2.

From these considerations we conclude that the third approximation, which is the second underestimated value of the critical load obtained by solving the cubic equation (41.33), provides a safety margin for the stability.

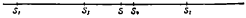

Fig. 41.2

A similar method of solving stability problems may be used for any other distribution of the axial loading if the rigidity of the bar also follows a certain law.

42. BEAMS ON ELASTIC FOUNDATION UNDER COMBINED COMPRESSION AND BENDING

Let beam EI of uniform rigidity be located on an elastic foundation of a linearly varying rigidity; suppose this beam is subjected to axial forces and to an arbitrary transverse load, as shown in Fig. 42.1. The rigidity of the foundation at section X with abscissa x is given by

$$k_X = k_0 + k_1 x \tag{42.1}$$

and the differential equation of the flexural curve is written according to the considerations outlined in Chap. VI and VII

$$EI\,\frac{d^4v}{dx^4} + S\,\frac{d^2v}{dx^2} + k_0 v + k_1 x v = M_{-1} \tag{42.2}$$

The following designations are introduced

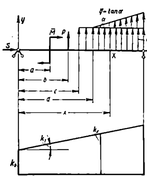

Fig. 42.1

$$S = \alpha^2 EI;\quad k_0 = \beta^2 EI;\quad k_1 = \gamma^2 EI \tag{42.3}$$

Then

$$EI\,\frac{d^4v}{dx^4} + \alpha^2 EI\,\frac{d^2v}{dx^2} + \beta^2 EIv + \gamma^2 xEIv = M_{-1} \tag{42.4}$$

This is a differential equation of the fourth order with a right-hand side and constant coefficients; it will be solved by the method of successive approximations.

The first approximation is derived by assuming that the axial force S and the elastic foundation are absent, i.e., $S = 0$ and $k_x = 0$. Then $\alpha = \beta = \gamma = 0$ and therefore

$$EI\,\frac{d^4v}{dx^4} = M_{-1}$$

$$EI\,\frac{d^3v}{dx^3} = M_0; \quad EI\,\frac{d^2v}{dx^2} = \frac{M_1}{1!}$$

$$EI\,\frac{dv}{dx} = \frac{M_2}{2!}; \quad EIv = \frac{M_3}{3!}$$

and every subsequent approximation is obtained by integrating equation

$$EI\,\frac{d^4v}{dx^4} = -\alpha^2 EI\,\frac{d^2v}{dx^2} - \beta^2 EIv - \gamma^2 EIxv \tag{42.5}$$

after substituting the values of $EI\,\frac{d^2v}{dx^2}$ and EIv obtained from the corresponding expressions determined in the previous approximation, as shown in Sec. 29.

By adding the results of successive approximations, we obtain in the fourth approximation the general expressions for Q, M, $EI\Theta$ and EIv.

For the transverse force

$$\begin{aligned}
Q = M_0 &- \left\{\alpha^2\frac{M_2}{2!} + \beta^2\frac{M_4}{4!} + \gamma^2\left[x\,\frac{M_4}{4!} - \frac{M_5}{5!}\right]\right\} + \\
&+ \left\{\alpha^4\frac{M_4}{4!} + 2\alpha^2\beta^2\frac{M_6}{6!} + 2\alpha^2\gamma^2\left[x\,\frac{M_6}{6!} - 2\,\frac{M_7}{7!}\right] + \right. \\
&+ 2\beta^2\gamma^2\left[x\,\frac{M_8}{8!} - 3\,\frac{M_9}{9!}\right] + \beta^4\frac{M_8}{8!} + \\
&\left. + 2\gamma^4\left[\frac{x^2}{2!}\cdot\frac{M_8}{8!} - 3x\,\frac{M_9}{9!} + 3\,\frac{M_{10}}{10!}\right]\right\} - \\
&- \left\{\alpha^6\frac{M_6}{6!} + 3\alpha^4\beta^2\frac{M_8}{8!} + 3\alpha^4\gamma^2\left[x\,\frac{M_8}{8!} - 3\,\frac{M_9}{9!}\right] + \right. \\
&+ 6\alpha^2\beta^2\gamma^2\left[x\,\frac{M_{10}}{10!} - 4\,\frac{M_{11}}{11!}\right] + 3\alpha^2\beta^4\frac{M_{10}}{10!} + \\
&+ 2\alpha^2\gamma^4\left[3\,\frac{x^2}{2!}\cdot\frac{M_{10}}{10!} - 12x\,\frac{M_{11}}{11!} + 19\,\frac{M_{12}}{12!}\right] + \\
&+ 3\beta^4\gamma^2\left[x\,\frac{M_{12}}{12!} - 5\,\frac{M_{13}}{13!}\right] + \beta^6\frac{M_{12}}{12!} - \\
&- 6\beta^2\gamma^4\left[\frac{x^2}{2!}\cdot\frac{M_{12}}{12!} - 5x\,\frac{M_{13}}{13!} + 11\,\frac{M_{14}}{14!}\right] + 6\gamma^6 \times \\
&\left. \times\left[\frac{x^3}{3!}\cdot\frac{M_{12}}{12!} - \frac{x^2}{2!}\cdot\frac{M_{13}}{13!} + 11x\,\frac{M_{14}}{14!} - 11\,\frac{M_{15}}{15!}\right]\right\}
\end{aligned} \tag{42.6}$$

For the bending moments

$$\begin{aligned} M = M_1 &- \left\{ \alpha^2 \frac{M_3}{3!} + \beta^2 \frac{M_5}{5!} + \gamma^2 \left[x \frac{M_5}{5!} - 2 \frac{M_6}{6!} \right] \right\} + \\ &+ \left\{ \alpha^4 \frac{M_5}{5!} + 2\alpha^2\beta^2 \frac{M_7}{7!} + 2\alpha^2\gamma^2 \left[x \frac{M_7}{7!} - 3 \frac{M_8}{8!} \right] + \right. \\ &+ 2\beta^2\gamma^2 \left[x \frac{M_9}{9!} - 4 \frac{M_{10}}{10!} \right] + \beta^4 \frac{M_9}{9!} + \\ &\left. + 2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_9}{9!} - 4x \frac{M_{10}}{10!} + 7 \frac{M_{11}}{11!} \right] \right\} - \\ &- \left\{ \alpha^6 \frac{M_7}{7!} + 3\alpha^4\beta^2 \frac{M_9}{9!} + 3\alpha^4\gamma^2 \left[x \frac{M_9}{9!} - 4 \frac{M_{10}}{10!} \right] - \right. \\ &- 6\alpha^2\beta^2\gamma^2 \left[x \frac{M_{11}}{11!} - 5 \frac{M_{12}}{12!} \right] + 3\alpha^2\beta^4 \frac{M_{11}}{11!} + \\ &+ 2\alpha^2\gamma^4 \left[3 \frac{x^2}{2!} \cdot \frac{M_{11}}{11!} - 15x \frac{M_{12}}{12!} + 34 \frac{M_{13}}{13!} \right] + \\ &+ 3\beta^4\gamma^2 \left[x \frac{M_{13}}{13!} - 6 \frac{M_{14}}{14!} \right] + \beta^6 \frac{M_{13}}{13!} + \\ &+ 6\beta^2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_{13}}{13!} - 6x \frac{M_{14}}{14!} + 17 \frac{M_{15}}{15!} \right] + \\ &\left. + 6\gamma^2 \left[\frac{x^3}{3!} \cdot \frac{M_{13}}{13!} - 6 \frac{x^2}{2!} \cdot \frac{M_{14}}{14!} + 17x \frac{M_{15}}{15!} - 28 \frac{M_{16}}{16!} \right] \right\} \end{aligned} \qquad (42.7)$$

For the rotation angles of sections

$$\begin{aligned} EI\Theta = \frac{M_2}{2!} &- \left\{ 2\alpha^2 \frac{M_4}{4!} + \beta^2 \frac{M_6}{6!} + \gamma^2 \left[x \frac{M_6}{6!} - 3 \frac{M_7}{7!} \right] \right\} + \\ &+ \left\{ \alpha^4 \frac{M_6}{6!} + 2\alpha^2\beta^2 \frac{M_8}{8!} + 2\alpha^2\gamma^2 \left[x \frac{M_8}{8!} - 4 \frac{M_9}{9!} \right] + \right. \\ &+ 2\beta^2\gamma^2 \left[x \frac{M_{10}}{10!} - 5 \frac{M_{11}}{11!} \right] + \beta^4 \frac{M_{10}}{10!} + 2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_{10}}{10!} - \right. \\ &\left. \left. - 5x \frac{M_{11}}{11!} + 12 \frac{M_{12}}{12!} \right] \right\} - \left\{ \alpha^6 \frac{M_8}{8!} + 3\alpha^4\beta^2 \cdot \frac{M_{10}}{10!} + \right. \\ &+ 3\alpha^4\gamma^2 \left[x \frac{M_{10}}{10!} - 5 \frac{M_{11}}{11!} \right] + 6\alpha^2\beta^2\gamma^2 \left[x \frac{M_{12}}{12!} - 6 \frac{M_{13}}{13!} \right] + \\ &+ 3\alpha^2\beta^4 \frac{M_{12}}{12!} + 2\alpha^2\gamma^4 \left[3 \frac{x^2}{2!} \cdot \frac{M_{12}}{12!} - 18x \frac{M_{13}}{13!} + \right. \\ &\left. + 52 \frac{M_{14}}{14!} \right] + 3\beta^4\gamma^2 \left[x \frac{M_{14}}{14!} - 7 \frac{M_{15}}{15!} \right] + \beta^6 \frac{M_{14}}{14!} + \\ &+ 6\beta^2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_{14}}{14!} - 7x \frac{M_{15}}{15!} + 24 \frac{M_{16}}{16!} \right] + 6\gamma^6 \left[\frac{x^3}{3!} \cdot \frac{M_{14}}{14!} - \right. \\ &\left. \left. - 7 \frac{x^2}{2!} \cdot \frac{M_{15}}{15!} + 24x \frac{M_{16}}{16!} - 52 \frac{M_{17}}{17!} \right] \right\} \end{aligned} \qquad (42.8)$$

For the deflections

$$\begin{aligned} EIv = {} & \frac{M_3}{3!} - \left\{ \alpha^2 \frac{M_5}{5!} + \beta^2 \frac{M_7}{7!} + \gamma^2 \left[x \frac{M_7}{7!} - 4 \frac{M_8}{8!} \right] \right\} + \\ & + \left\{ \alpha^4 \frac{M_7}{7!} + 2\alpha^2\beta^2 \frac{M_9}{9!} + 2\alpha^2\gamma^2 \left[x \frac{M_9}{9!} - 5 \frac{M_{10}}{10!} \right] + \right. \\ & + 2\beta^2\gamma^2 \left[x \frac{M_{11}}{11!} - 6 \frac{M_{12}}{12!} \right] + \beta^4 \frac{M_{11}}{11!} + 2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_{11}}{11!} - \right. \\ & \left. \left. - 6x \frac{M_{12}}{12!} + 18 \frac{M_{13}}{13!} \right] \right\} - \left\{ \alpha^6 \frac{M_9}{9!} + 3\alpha^4\beta^2 \frac{M_{11}}{11!} + \right. \\ & + 3\alpha^4\gamma^2 \left[x \frac{M_{11}}{11!} - 6 \frac{M_{12}}{12!} \right] + 6\alpha^2\beta^2\gamma^2 \left[x \frac{M_{13}}{13!} - 7 \frac{M_{14}}{14!} \right] + \\ & + 3\alpha^2\beta^4 \frac{M_{13}}{13!} + 2\alpha^2\gamma^4 \left[3 \frac{x^2}{2!} \cdot \frac{M_{13}}{13!} - 21x \frac{M_{14}}{14!} + 73 \frac{M_{15}}{15!} \right] + \\ & + 3\beta^4\gamma^2 \left[x \frac{M_{15}}{15!} - 8 \frac{M_{16}}{16!} \right] + \beta^6 \frac{M_{15}}{15!} + 6\beta^2\gamma^4 \left[\frac{x^2}{2!} \cdot \frac{M_{15}}{15!} - \right. \\ & \left. - 8x \frac{M_{16}}{16!} + 32 \frac{M_{17}}{17!} \right] + 6\gamma^6 \left[\frac{x^3}{3!} \cdot \frac{M_{15}}{15!} - 8 \frac{x^2}{2!} \cdot \frac{M_{16}}{16!} + \right. \\ & \left. \left. + 32x \frac{M_{17}}{17!} - 84 \frac{M_{18}}{18!} \right] \right\} \qquad (42.9) \end{aligned}$$

Formulas (42.6), (42.8) and (42.9) are of a general type, because they may be applied to beams on elastic foundation, both with or without rigid supports and to beams subjected to any transverse load and a constant load S. The following particular cases are derived from these formulas:

(a) $\alpha = 0$ —a beam on elastic foundation of rigidity $k_x = k_0 + k_1 x$;

(b) $\alpha = 0$, $\beta = 0$ — a beam on elastic foundation of rigidity $k_x = k_1 x$;

(c) $\alpha \neq 0$, $\beta = 0$, $\gamma = 0$ — an ordinary beam under combined compression and flexure;

(d) $\alpha = 0$, $\beta = 0$, $\gamma = 0$ — a beam under an arbitrary transverse load.

Formulas (42.6) to (42.9) give the solution of the fourth approximation. The first, second and third approximations are obtained from these formulas by taking one, two, and three terms, respectively, in each formula.

V. Kopylenko solved several practical problems to the fourth approximation: their comparison with exact solutions showed negligible deviations.

N. Soldatov showed that the infinite series obtained by the moment-operational method represented MacLaurin's series of Puzyrevsky's function for the classical solution of a beam resting on an elastic foundation. N. Soldatov examined the results and came to the conclusion that the solution obtained by V. Kopylenko in the third approximation is equal to the exact solution given by the classic theory (up to the seventh figure). However, for a more rigid and a longer foundation the fourth approximation may be necessary. In general, every particular loading diagram requires a special evaluation on the basis of the Tailor-MacLaurin's power series, because the sum of the rejected terms does not exceed the last remaining term of the series. Thus, the exactness of the obtained results may be found by determining the value of the last term of the series.

43. NON-UNIFORM BEAM UNDER AXIAL FORCE

Consider a function $y = F(x)$ which is a product of two functions

$$y = \varphi(x)\, f(x) \tag{43.1}$$

the first of which is a limited and continuous function, while the second one is a sum of a finite number of functions, which are equal to zero on the left-hand side of a certain point and continuous on the right-hand side, being either continuous or discontinuous at the point itself. Let φ', φ'', φ''', . . . designate the first, second, third,... derivatives of φ with respect to x and f_1, f_2, f_3, . . . designate the first, second, third,... integrals of $f(x)$.

Let $I_{1,1}$, $I_{2,1}$, $I_{3,1}$, . . . designate the first, second, third,... integrals of $y = \varphi(x) f(x)$ and

$$I_{n,1} = \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi f\, dx^n \tag{43.2}$$

represent the product of f and φ integrated n times.

Similarly, $I_{n,2}$ will represent a double product of f and φ integrated n times

$$I_{n,2} = \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi I_{n,1}\, dx^n = \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi\, dx^n \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi f\, dx^n \tag{43.3}$$

In other words, $I_{n,2}$ is determined by multiplying $f(x)$ by $\varphi(x)$, integrating the result n times, then multiplying the integral by $\varphi(x)$ the second time and integrating the result n times more.

Similarly, the following integral

$$I_{n,3} = \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi\, dx^n \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi\, dx^n \underbrace{\int \cdots \int}_{n\ \text{times}} \varphi f\, dx^n \tag{43.4}$$

represents the result of applying the described procedure three times: first, function $f(x)$ is multiplied by $\varphi(x)$ and the result is integrated

n times, then this integral is again multiplied by $\varphi(x)$ and the result integrated n times more, and, finally, the result is multiplied by $\varphi(x)$ for the third time and then integrated n times.

Thus, the function $y = \varphi(x) f(x)$ is calculated by partial integration

$$I_{1,1} = \varphi f_1 - \varphi' f_2 + \varphi'' f_3 - \varphi''' f_4 + \varphi^{IV} f_5 - \dots \tag{43.5}$$

$$I_{2,1} = \varphi f_2 - 2\varphi' f_3 + 3\varphi'' f_4 - 4\varphi''' f_5 + 5\varphi^{IV} f_6 - \dots \tag{43.6}$$

$$I_{3,1} = \varphi f_3 - 3\varphi' f_4 + 6\varphi'' f_5 - 10\varphi''' f_6 + 15\varphi^{IV} f_7 - \dots \tag{43.7}$$

. . . .

These integrals may be expressed by a general formula (assuming that $\varphi^{(0)} = \varphi$)

$$I_{n,1} = \sum_{k=1,2\dots} (-1)^{k-1} \frac{(k+n-2)!}{(k-1)!\,(n-1)!} \varphi^{(k-1)} f_{k+n-1} \tag{43.8}$$

Coefficients for various values of n (number of integrations) and k (ordinal number of term in the series) are given in Table 43.1.

Table 43.1

Coefficients for Various Values of n and k

n	k					n	k				
	1	2	3	4	5		1	2	3	4	5
1	1	1	1	1	1	6	1	6	21	56	126
2	1	2	3	4	5	7	1	7	28	84	210
3	1	3	6	10	15	8	1	8	36	120	330
4	1	4	10	20	35	9	1	9	45	165	495
5	1	5	15	35	70	10	1	10	55	220	715

According to Pascal's triangle, every coefficient of this table is a sum of two coefficients preceding it in column and in line. This relation may be given in a general form by examining formula (43.8): from this formula it follows that

$$u_{n,2k} = \frac{(2k+n-2)!}{(2k-1)!\,(n-1)!}; \quad u_{n-1,2k+1} = \frac{(2k+n-2)!}{2k!\,(n-2)!}$$

and the sum of these coefficients which precede the coefficient $u_{n,2k+1}$ in column and in line is

$$u_{n,2k} + u_{n-1,2k+1} = u_{n,2k+1} \tag{43.9}$$

Let us calculate the following integral

$$I_{2,2} = \int\int \varphi\,dx^2 \int\int \varphi f\,dx^2 = \int\int \varphi I_{2,1}\,dx^2 \tag{43.10}$$

If we substitute $I_{2,1}$ in (43.6) for f successive partial integration gives the following formula

$$I_{2,2}=\varphi i_2(I_{2,1})-2\varphi' i_3(I_{2,1})+3\varphi'' i_4(I_{2,1})-$$
$$-4\varphi''' i_5(I_{2,1})+\varphi^{IV} i_6(I_{2,1})-\dots$$

where $i_2(I_{2,1})$, $i_3(I_{2,1})$, ... are the second, third,... integrals of $I_{2,1}$. However, according to the integration procedure

$$i_2(I_{2,1})=I_{4,1};\quad i_3(I_{2,1})=I_{5,1}\ \dots$$

Hence,

$$I_{2,2}=\varphi I_{4,1}-2\varphi' I_{5,1}+3\varphi'' I_{6,1}-4\varphi'' I_{7,1}+5\varphi^{IV} I_{8,1}-\dots \qquad (43.11)$$

Similarly, the expression for $I_{2,3}$ is given by

$$I_{2,3}=\int\int \varphi I_{2,2}\,dx^2=\varphi i_2(I_{2,2})-2\varphi' i_3(I_{2,2})+$$
$$+3\varphi'' i_4(I_{2,2})-4\varphi''' i_5(I_{2,2})+5\varphi^{IV} i_6(I_{2,2})-\dots \qquad (43.12)$$

The quantities contained in formula (43.12) are equal to

$$i_2(I_{2,2})=\int\int I_{2,2}\,dx^2=\int\int\int\int \varphi I_{2,1}\,dx^4=$$
$$=\varphi i_4(I_{2,1})-4\varphi' i_5(I_{2,1})+10\varphi'' i_6(I_{2,1})-$$
$$-20\varphi''' i_7(I_{2,1})+35\varphi^{IV} i_8(I_{2,1})-\dots=\varphi I_{6,1}-$$
$$-4\varphi' I_{7,1}+10\varphi'' I_{8,1}-20\varphi''' I_{9,1}+35\varphi^{IV} I_{10,1}-\dots \qquad (43.13)$$

$$i_3(I_{2,2})=\int\int\int I_{2,2}\,dx^3=i_5(\varphi I_{2,1})=$$
$$=\varphi I_{7,1}-5\varphi' I_{8,1}+15\varphi'' I_{9,1}-35\varphi'' I_{10,1}+70\varphi^{IV} I_{11,1}-\dots \qquad (43.14)$$

$$i_4(I_{2,2})=\int\int\int\int I_{2,2}\,dx^4=i_6(\varphi I_{2,1})=$$
$$=\varphi i_6(I_{2,1})-6\varphi' i_7(I_{2,1})+21\varphi'' i_8(I_{2,1})-56\varphi''' i_9(I_{2,1})+$$
$$+126\varphi^{IV} i_{10}(I_{2,1})-\dots=\varphi I_{8,1}-6\varphi' I_{9,1}+$$
$$+21\varphi'' I_{10,1}-56\varphi'' I_{11,1}+126\varphi^{IV} I_{12,1}-\dots \qquad (43.15)$$

In the same way we obtain the fifth and sixth integrals of the function $I_{2,2}$

$$i_5(I_{2,2})=i_7(\varphi I_{2,1})=\varphi I_{9,1}-7\varphi' I_{10,1}+$$
$$+28\varphi'' I_{11,1}-84\varphi''' I_{12,1}+210\varphi^{IV} I_{13,1}-\dots \qquad (43.16)$$

$$i_6(I_{2,2})=i_8(\varphi I_{2,1})=\varphi I_{10,1}-8\varphi' I_{11,1}+$$
$$+36\varphi'' I_{12,1}-120\varphi''' I_{13,1}+330\varphi^{IV} I_{14,1}-\dots \qquad (43.17)$$

Similarly, $I_{2,4}$ can be presented as

$$I_{2,4}=\int\int \varphi I_{2,3}\,dx^2=\varphi i_2(I_{2,3})-2\varphi' i_3(I_{2,3})+$$
$$+3\varphi'' i_4(I_{2,3})-4\varphi''' i_5(I_{2,3})+5\varphi^{IV} i_6(I_{2,3})-\dots \qquad (43.18)$$

where

$$i_2(I_{2,3})=i_4(\varphi I_{2,2})=\varphi i_4(I_{2,2})-4\varphi' i_5(I_{2,2})+$$
$$+10\varphi'' i_6(I_{2,2})-20\varphi''' i_7(I_{2,2})+35\varphi^{IV} i_8(I_{2,2})-\dots,$$

By substituting into (43.11) the expressions for integral quantities $I_{4,1}, I_{5,1}\dots$ given by (43.8) and considering Table 43.1 we obtain

$$I_{2,2}=\varphi^2 f_4-6\varphi\varphi' f_5+(13\varphi\varphi''+10\varphi'^2)f_6-$$
$$-(24\varphi\varphi'''+48\varphi'\varphi'')f_7+\dots \qquad (43.19)$$

In order to apply the formulas derived above to beams of variable section subjected to compression and bending let us solve the following differential equation of the flexural curve

$$EI\frac{d^2v}{dx^2}+Sv=M_1 \qquad (43.20)$$

by successive approximations.

In (43.20) the quantity S represents the longitudinal compressive force and M_1 is the bending moment caused by the transversal load.

Assuming for the first approximation that $S=0$, we obtain

$$EI\frac{d^2v}{dx^2}=M_1$$

Designating

$$\frac{1}{EI}=\gamma$$

we arrive at

$$\frac{d^2v}{dx^2}=\gamma M_1$$

Hence,

$$v=\int\int \gamma M_1\,dx^2 \qquad (43.21)$$

The second approximation is derived from (43.20) by considering the following equation

$$EI\frac{d^2v}{dx^2}=-Sv$$

which may be transformed by substitution of (43.21) into

$$EI\frac{d^2v}{dx^2}=-S\int\int \gamma M_1\,dx^2$$
$$\frac{d^2v}{dx^2}=-S\gamma\int\int \gamma M_1\,dx^2$$

and, finally,

$$v = -S \int\int \gamma\, dx^2 \int\int \gamma M_1\, dx^2 \tag{43.22}$$

Substituting $\gamma = \varphi$ and $M_1 = f$, we have

$$v = I_{2,\,1} - S I_{2,\,2} + S^2 I_{2,\,3} - \dots \tag{43.23}$$

in which the integral quantities $I_{2,\,1}$, $I_{2,\,2}$, $I_{2,\,3}$ . . . are determined from the above mentioned formulas.

For practical purposes two terms are sufficient in evaluating strength and rigidity of non-uniform beams under combined compression and bending

$$v = I_{2,\,1} - S I_{2,\,2} \tag{43.24}$$

and the magnitude obtained is a little larger than the exact value, which is of considerable practical importance.

Assuming $\varphi = \gamma$ and $f = M_1$ in formulas (43.6) and (43.19) we obtain an expression for substitution into formula (43.24)

$$\begin{aligned} v = & \left[\gamma \frac{M_3}{3!} - 2\gamma' \frac{M_4}{4!} + 3\gamma'' \frac{M_5}{5!} - 4\gamma''' \frac{M_6}{6!} + 5\gamma^{\mathrm{IV}} \frac{M_7}{7!} - \dots\right] - \\ & - S\left[\gamma^2 \frac{M_5}{5!} - 6\gamma\gamma' \frac{M_6}{6!} + (13\gamma\gamma'' + 10\gamma'^2) \frac{M_7}{7!} - \right. \\ & \left. - (24\gamma\gamma''' + 48\gamma'\gamma'') \frac{M_8}{8!} + \dots\right] \end{aligned} \tag{43.25}$$

Higher moments entering this formula should be determined from the basic formula (I) of this monograph; as to the derivatives of the expressions for the flexibility of the beam (γ', γ''), they are especially easy to calculate when the flexibility, which is inverse to rigidity, is expressed (either exactly or approximately) by an algebraic polinomial

$$\gamma = \gamma_0 + \gamma_1 x + \gamma_2 \frac{x^2}{2} + \gamma_3 \frac{x^3}{3!} + \dots \tag{43.26}$$

In this particular case

$$\begin{aligned} \gamma' &= \gamma_1 + \gamma_2 x + \gamma_3 \frac{x^2}{2} + \dots \\ \gamma'' &= \gamma_2 + \gamma_3 x + \dots \\ \gamma''' &= \gamma_3 + \gamma_4 x + \dots \text{ etc.} \end{aligned}$$

Problem 1. Let a beam, fixed at one end, be subjected to a point load P and an axial force S (Fig. 43.1) and the flexibility $\gamma = \frac{1}{EI}$ of the beam varies according to a linear law.

Let $\gamma_0 = \frac{1}{EI_0}$ represent the flexibility of the beam at the fixed end; then for an arbitrary section the following expressions may be derived

$$\left.\begin{aligned} i &= \frac{EI_0}{EI} = m - \frac{m-1}{l}\,x \\ \gamma &= \gamma_0 i = \gamma_0\left(m - \frac{m-1}{l}\,x\right) \\ \gamma' &= -\gamma_0\frac{m-1}{l};\ \gamma'' = 0;\ \gamma''' = 0,\ \text{etc.} \end{aligned}\right\} \tag{43.27}$$

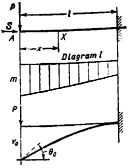

Fig. 43.1

The flexibility and its first derivative at the fixed end (when $x = l$) are equal to

$$\gamma = \gamma_0 \text{ and } \gamma' = -\gamma_0\frac{m-1}{l} \tag{43.28}$$

Substituting $\varphi = \gamma$ and $f = M_1$ into the general formulas, we obtain from formulas (43.6) and (43.19) the following expressions

$$I_{2,1} = \gamma\,\frac{M_3}{3!} - 2\gamma'\,\frac{M_4}{4!} \tag{43.29}$$

$$I_{2,2} = \gamma I_{4,1} - 2\gamma' I_{5,1} \tag{43.30}$$

and from formula (43.8) we have

$$I_{4,1} = \gamma\,\frac{M_5}{5!} - 4\gamma'\,\frac{M_6}{6!} \tag{43.31}$$

$$I_{5,1} = \gamma\,\frac{M_6}{6!} - 5\gamma'\,\frac{M_7}{7!} \tag{43.32}$$

which may be substituted into (43.30) and give

$$I_{2,2} = \gamma^2\,\frac{M_5}{5!} - 6\gamma\gamma'\,\frac{M_6}{6!} + 10\gamma'^2\,\frac{M_7}{7!} \tag{43.33}$$

This expression could be also derived from formula (43.19). Similarly, according to formula (43.12), we obtain

$$\begin{aligned} I_{2,3} &= \iint \gamma I_{2,2}\,dx^2 = \gamma i_2(I_{2,2}) - 2\gamma' i_3(I_{2,2}) = \\ &= \gamma(\gamma I_{6,1} - 4\gamma' I_{7,1}) - 2\gamma'(\gamma I_{7,1} - 5\gamma' I_{8,1}) = \\ &= \gamma^2 I_{6,1} - 6\gamma\gamma' I_{7,1} + 10\gamma'^2 I_{8,1} \end{aligned}$$

However, it follows from Table 43.1 and formula (43.8) that

$$I_{6.1}=\gamma\frac{M_7}{7!}-6\gamma'\frac{M_8}{8!}$$

$$I_{7.1}=\gamma\frac{M_8}{8!}-7\gamma'\frac{M_9}{9!}$$

and

$$I_{8.1}=\gamma\frac{M_9}{9!}-8\gamma'\frac{M_{10}}{10!}$$

Hence,

$$I_{2.3}=\gamma^3\frac{M_7}{7!}-12\gamma^2\gamma'\frac{M_8}{8!}+52\gamma\gamma'^2\frac{M_9}{9!}-80\gamma'^3\frac{M_{10}}{10!} \qquad (43.34)$$

Substitution of $I_{2.1}$, $I_{2.2}$ and $I_{2.3}$ from formulas (43.29), (43.33) and (43.34), respectively, into formula (43.23) gives

$$\begin{aligned} v=&\left(\gamma\frac{M_3}{3!}-2\gamma'\frac{M_4}{4!}\right)-S\left(\gamma^2\frac{M_5}{5!}-6\gamma\gamma'\frac{M_6}{6!}+\right.\\ &\left.+10\gamma'^2\frac{M_7}{7!}\right)+S^2\left(\gamma^3\frac{M_7}{7!}-12\gamma^2\gamma'\frac{M_8}{8!}+\right.\\ &\left.+52\gamma\gamma'^2\frac{M_9}{9!}-80\gamma'^3\frac{M_{10}}{10!}\right)-\dots \end{aligned} \qquad (43.35)$$

In this formula the higher moments, the flexibility and its derivative are functions of the abscissa; by differentiating this expression with respect to x and considering the relation $\frac{d}{dx}\cdot\frac{M_n}{n!}=\frac{M_{n-1}}{(n-1)!}$ and formula (43.27) we find the expression for the angle of rotation

$$\begin{aligned} \Theta=&\left(\gamma\frac{M_2}{2!}-\gamma'\frac{M_3}{3!}\right)-S\left(\gamma^2\frac{M_4}{4!}-4\gamma\gamma'\frac{M_5}{5!}+\right.\\ &\left.+4\gamma'^2\frac{M_6}{6!}\right)+S^2\left(\gamma^3\frac{M_6}{6!}-9\gamma^2\gamma'\frac{M_7}{7!}+\right.\\ &\left.+28\gamma\gamma'^2\frac{M_8}{8!}-28\gamma'^3\frac{M_9}{9!}\right)-\dots \end{aligned} \qquad (43.36)$$

Further differentiation will give the expression for the curvature k. However, $k=\frac{M}{EI}=\gamma M$, hence,

$$\begin{aligned} \gamma M=&\gamma M_1-S\left(\gamma^2\frac{M_3}{3!}-2\gamma\gamma'\frac{M_4}{4!}\right)+\\ &+S^2\left(\gamma^3\frac{M_5}{5!}-6\gamma^2\gamma'\frac{M_6}{6!}+10\gamma\gamma'^2\frac{M_7}{7!}\right)-\dots \end{aligned} \qquad (43.37)$$

or, after dividing both sides of this formula by γ

$$\begin{aligned} M=&M_1-S\left(\gamma\frac{M_3}{3!}-2\gamma'\frac{M_4}{4!}\right)+\\ &+S^2\left(\gamma^2\frac{M_5}{5!}-6\gamma\gamma'\frac{M_6}{6!}+10\gamma'^2\frac{M_7}{7!}\right)-\dots \end{aligned} \qquad (43.38)$$

Further differentiation will give the expressions for the transversal force and its derivative

$$Q=\frac{dM}{dx}=M_0-S\left(\gamma\frac{M_2}{2!}-\gamma'\frac{M_3}{3!}\right)+$$
$$+S^2\left(\gamma^2\frac{M_4}{4!}-4\gamma\gamma'\frac{M_5}{5!}+4\gamma'^2\frac{M_6}{6!}\right)-\dots \qquad (43.39)$$

and

$$\frac{d^2M'}{dx^2}=M_{-1}-S\gamma M_1+S^2\left(\gamma^2\frac{M_3}{3!}-2\gamma\gamma'\frac{M_4}{4!}\right)-\dots \qquad (43.40)$$

Formulas (43.35), (43.36) and (43.38)-(43.40) allow the corresponding quantities of any compound load consisting of point and distributed forces and force couples to be determined on the basis of general formula (I). As for the kinematic parameters (in particular, the initial values of rotation angles and linear displacements), they require special consideration. Assuming that no static load (force or moment) is applied to the beam, from formula (43.20) we obtain

$$\left.\begin{aligned} &EI\frac{d^2v}{dx^2}+Sv=0,\ EI\frac{d^2v}{dx^2}=-Sv \\ \text{or} \\ &\frac{d^2v}{dx^2}=-\frac{1}{EI}Sv;\ \frac{d^2v}{dx^2}=-\gamma Sv \end{aligned}\right\} \qquad (43.41)$$

Assuming for the first approximation that

$$\frac{d^2v}{dx^2}=-\gamma SM_0(v) \qquad (43.42)$$

we derive by integration, according to formulas (43.5) and (43.6)

$$\frac{dv}{dx}=-S\int\gamma M_0(v)\,dx=-S\left[\gamma M_1-\gamma'\frac{M_2}{2!}\right](v) \qquad (43.43)$$

$$v=-S\int\int\gamma M_0(v)\,dx^2=-S\left[\gamma\frac{M_2}{2!}-2\gamma'\frac{M_3}{3!}\right](v) \qquad (43.44)$$

By substituting (43.44) into (43.41) we arrive at

$$\frac{d^2v}{dx^2}=+S^2\left[\gamma^2\frac{M_2}{2!}-2\gamma\gamma'\frac{M_3}{3!}\right](v) \qquad (43.45)$$

The second integral of this equation gives the expression for v; the same result may be obtained directly from formula

$$v=S^2\int\int\gamma\,dx^2\int\int\gamma M_0\,dx^2 \qquad (43.46)$$

from which, according to formula (43.23), we find

$$v=S^2\left(\gamma^2\frac{M_4}{4!}-6\gamma\gamma'\frac{M_5}{5!}+10\gamma'^2\frac{M_6}{6!}\right)(v) \qquad (43.47)$$

Term by term differentiation of (43.47) gives us

$$\Theta = S^2\left(\gamma^2\frac{M_3}{3!} - 4\gamma\gamma'\frac{M_4}{4!} + 4\gamma'^2\frac{M_5}{5!}\right)(v) \tag{43.48}$$

Combining formulas (43.42), (43.44) and (43.47) we get the third approximation for the deflection caused by the linear kinematic load

$$\begin{aligned} v = \Big\{ M_0 - S\left[\gamma\frac{M_2}{2!} - 2\gamma'\frac{M_3}{3!}\right] + \\ + S^2\left[\gamma^2\frac{M_4}{4!} - 6\gamma\gamma'\frac{M_5}{5!} + 10\gamma'^2\frac{M_6}{6!}\right] - \ldots \Big\}(v) \end{aligned} \tag{43.49}$$

while combining formulas (43.43) and (43.48) we get the third approximation for the angle of rotation caused by the same kinematic load

$$\begin{aligned} \Theta = \Big\{ -S\left[\gamma M_1 - \gamma'\frac{M_2}{2!}\right] + \\ + S^2\left[\gamma^2\frac{M_3}{3!} - 4\gamma\gamma'\frac{M_4}{4!} + 4\gamma'^2\frac{M_5}{5!}\right] - \ldots \Big\}(v) \end{aligned} \tag{43.50}$$

Formulas for deflections and rotation angles due to angular kinematic loads are integral expressions with respect to the corresponding formulas referring to the linear kinematic loads; hence, from (43.49) and (43.50) the following is derived

$$\begin{aligned} v = \Big\{ M_1 - S\left[\gamma\frac{M_3}{3!} - 2\gamma'\frac{M_4}{4!}\right] + \\ + S^2\left[\gamma^2\frac{M_5}{5!} - 6\gamma\gamma'\frac{M_6}{6!} + 10\gamma'^2\frac{M_7}{7!}\right] - \ldots \Big\}(\Theta) \end{aligned} \tag{43.51}$$

$$\begin{aligned} \Theta = \Big\{ M_0 - S\left[\gamma\frac{M_2}{2!} - \gamma'\frac{M_3}{3!}\right] + \\ + S^2\left[\gamma^2\frac{M_4}{4!} - 4\gamma\gamma'\frac{M_5}{5!} + 4\gamma'^2\frac{M_6}{6!}\right] - \ldots \Big\}(\Theta) \end{aligned} \tag{43.52}$$

When no kinematic loads are applied, the formulas derived above can be used to obtain the initial angle Θ_A and the initial deflection v_A at section A. For an arbitrary section X with abscissa x the following is derived from formulas (43.49)-(43.52)

$$\begin{aligned} v = v_0\Big[1 - S\left(\gamma\frac{x^2}{2} - 2\gamma'\frac{x^3}{3!}\right) + S^2\Big(\gamma^2\frac{x^4}{4!} - \\ - 6\gamma\gamma'\frac{x^5}{5!} + 10\gamma'^2\frac{x^6}{6!}\Big) - \ldots\Big] + \Theta_0\Big[x - S\left(\gamma\frac{x^3}{3!} - 2\gamma'\frac{x^4}{4!}\right) + \\ + S^2\left(\gamma^2\frac{x^6}{5!} - 6\gamma\gamma'\frac{x^6}{6!} + 10\gamma'^2\frac{x^7}{7!}\right) - \ldots\Big] \end{aligned} \tag{43.53}$$

$$\Theta = v_0\Big[-S\left(\gamma x-\gamma'\frac{x^2}{2}\right)+S^2\left(\gamma^2\frac{x^3}{3!}-4\gamma\gamma'\frac{x^4}{4!}+\right.$$
$$\left.+4\gamma'^2\frac{x^5}{5!}\right)-\dots\Big]+\Theta_0\Big[1-S\left(\gamma\frac{x^2}{2!}-\gamma'\frac{x^3}{3!}\right)+$$
$$+S^2\left(\gamma^2\frac{x^4}{4!}-4\gamma\gamma'\frac{x^5}{5!}+4\gamma'^2\frac{x^6}{6!}\right)-\dots\Big] \tag{43.54}$$

Combination of formulas (43.35) and (43.53) gives an exact formula for the deflections, while the combination of formulas (43.36) and (43.54) gives an exact formula for angles of rotation. The two unknown values, Θ_0 and v_0, can be determined by considering (43.28), under the condition that at the right support ($x = l$) the linear and angular deflections are equal to zero

$$\Big\{\Big[\gamma_0\left(-\frac{Pl^3}{3!}\right)+2\gamma_0\frac{m-1}{l}\left(-\frac{Pl^4}{4!}\right)\Big]-S\Big[\gamma_0^2\left(-\frac{Pl^5}{5!}\right)+$$
$$+6\gamma_0^2\frac{m-1}{l}\left(-\frac{Pl^6}{6!}\right)+10\gamma_0^2\frac{(m-1)^2}{l^2}\left(-\frac{Pl^7}{7!}\right)\Big]+$$
$$+S^2\Big[\gamma_0^3\left(-\frac{Pl^7}{7!}\right)-12\gamma_0^2\left(-\gamma_0\frac{m-1}{l}\right)\left(-\frac{Pl^8}{8!}\right)+$$
$$+52\gamma_0\left(\gamma_0\frac{m-1}{l}\right)^2\left(-\frac{Pl^9}{9!}\right)-80\left(-\gamma_0\frac{m-1}{l}\right)^3\times$$
$$\times\left(-\frac{Pl^{10}}{10!}\right)\Big]-\dots\Big\}+v_0\Big\{1-S\Big[\gamma_0\frac{l^2}{2}-$$
$$-2\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^3}{3!}\Big]+S^2\Big[\gamma_0^2\frac{l^4}{4!}-6\gamma_0^2\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^5}{5!}+$$
$$+10\left(-\gamma_0\frac{m-1}{l}\right)^2\frac{l^6}{6!}\Big]-\dots\Big\}+$$
$$+\Theta_0\Big\{l-S\Big[\gamma_0\frac{l^3}{3!}-2\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^4}{4!}\Big]+$$
$$+S^2\Big[\gamma_0^2\frac{l^5}{5!}-6\gamma_0\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^6}{6!}+$$
$$+10\left(-\gamma_0\frac{m-1}{l}\right)^2\frac{l^7}{7!}\Big]-\dots\Big\}=0 \tag{43.55}$$

$$\Big\{\Big[\gamma_0\left(-\frac{Pl^2}{2}\right)-\left(-\gamma_0\frac{m-1}{l}\right)\frac{Pl^3}{3!}\Big]-S\Big[\gamma_0\left(-\frac{Pl^4}{4!}\right)-$$
$$-4\gamma_0\left(-\gamma_0\frac{m-1}{l}\right)\left(-\frac{Pl^5}{5!}\right)+$$
$$+4\left(-\gamma_0\frac{m-1}{l}\right)^2\left(-\frac{Pl^6}{6!}\right)\Big]+$$
$$+S^2\Big[\gamma_0^3\left(-\frac{Pl^6}{6!}\right)-9\gamma_0^2\left(-\gamma_0\frac{m-1}{l}\right)\left(-\frac{Pl^7}{7!}\right)+$$
$$+28\gamma_0\left(-\gamma_0\frac{m-1}{l}\right)^2\left(-\frac{Pl^8}{8!}\right)-$$

$$
\begin{aligned}
&-28\left(-\gamma_0\frac{m-1}{l}\right)^3\left(-\frac{Pl^9}{9!}\right)\Big]-\dots\Big\}+\\
&+v_0\Big\{-S\left[\gamma_0 l-\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^2}{2}\right]+\\
&+S^2\left[\gamma_0^2\frac{l^3}{3!}-4\gamma_0\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^4}{4!}+\right.\\
&\left.+4\left(-\gamma_0\frac{m-1}{l}\right)^2\frac{l^5}{5!}\right]-\dots\Big\}+\\
&+\Theta_0\Big\{1-S\left[\gamma_0\frac{l^2}{2}-\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^3}{3!}\right]+\\
&+S^2\left[\gamma_0^2\frac{l^4}{4!}-4\gamma_0\left(-\gamma_0\frac{m-1}{l}\right)\frac{l^5}{5!}+\right.\\
&\left.+4\left(-\gamma_0\frac{m-1}{l}\right)^2\frac{l^6}{6!}\right]-\dots\Big\}=0
\end{aligned}
\tag{43.56}
$$

By eliminating the terms containing S^2 in the last two formulas we obtain the second approximations of the values v_0 and Θ_0, which are slightly larger than the exact solution, and so are all the other quantities related to strength and rigidity of the given non-uniform beam subjected to combined compression and bending under forces P and S. In the case when any other transversal loads are applied to the beam, the only changes in the calculations will consist in introducing to (43.35) and (43.36) the expressions for higher moments corresponding to the particular type of loading.

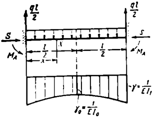

Fig. 43.2

Problem 2. Let a beam fixed at both ends be subjected to axial forces S and a uniform continuous load of intensity q (Fig. 43.2). The flexibility of the beam changes according to the second order parabola law

$$\gamma=\gamma_0\left[1+\left(1-2\frac{x}{l}\right)^2\frac{I_0-I_1}{I_1}\right]=\gamma_0\left[1+\alpha\left(1-2\frac{x}{l}\right)^2\right] \tag{43.57}$$

where $\gamma_0=\frac{1}{EI_0}$ = flexibility of the beam at mid-span

I_0 and I_1 = moments of inertia of the beam cross section at mid-span and supports, respectively

γ = flexibility of the beam at an arbitrary section with abscissa x

Then the derivatives of the flexibility are expressed by

$$\gamma' = -\gamma_0 \frac{4\alpha}{l}\left(1 - 2\frac{x}{l}\right); \quad \gamma'' = \gamma_0 \frac{8\alpha}{l^2}; \quad \gamma''' = 0 \qquad (43.58)$$

For the mid-span section

$$\gamma = \gamma_0; \quad \gamma' = 0; \quad \gamma'' = \gamma_0 \frac{4\alpha}{l^2}; \quad \gamma''' = 0 \qquad (43.59)$$

In order to apply formula (43.23), we must first determine the integral quantities $I_{2,1}$, $I_{2,2}$ and $I_{2,3}$. Assuming in (43.6) that $\varphi = \gamma$ and $f = M_1$ we find

$$I_{2,1} = \gamma \frac{M_3}{3!} - 2\gamma' \frac{M_4}{4!} + 3\gamma'' \frac{M_5}{5!} \qquad (43.60)$$

Similarly, from formula (43.19) we get

$$I_{2,2} = \gamma^2 \frac{M_5}{5!} - 6\gamma\gamma' \frac{M_6}{6!} + (13\gamma\gamma'' + 10\gamma'^2)\frac{M_7}{7!} -$$
$$-(24\gamma\gamma''' + 48\gamma'\gamma'')\frac{M_8}{8!} - \dots \qquad (43.61)$$

And, finally, from formula (43.12), with expressions (43.13)-(43.15) taken into account, we obtain

$$I_{2,3} = \int\int \gamma I_{2,2}\,dx^2 = \gamma i_2(I_{2,2}) - 2\gamma' i_3(I_{2,2}) + 3\gamma'' i_4(I_{2,2}) -$$
$$- \dots = \gamma(\gamma I_{6,1} + 4\gamma' I_{7,1} + 10\gamma'' I_{8,1}) - 2\gamma'(\gamma I_{7,1} - 5\gamma' I_{8,1} +$$
$$+ 15\gamma'' I_{9,1}) + 3\gamma''(\gamma I_{8,1} - 6\gamma' I_{9,1} + 21\gamma'' I_{10,1}) - \dots =$$
$$= \gamma^2 I_{6,1} - 6\gamma\gamma' I_{7,1} + (13\gamma\gamma'' + 10\gamma'^2) I_{8,1} - 48\gamma'\gamma'' I_{9,1} +$$
$$+ 63\gamma''^2 I_{10,1} - \dots \qquad (43.62)$$

However, according to Table 43.1 and formula (43.12), integrals $I_{6,1}$, $I_{7,1}$, $I_{8,1}$, $I_{9,1}$ and $I_{10,1}$ may be expressed in terms of higher moments

$$\left.\begin{aligned}
I_{6,1} &= \gamma \frac{M_7}{7!} - 6\gamma' \frac{M_8}{8!} + 21\gamma'' \frac{M_9}{9!} - \dots \\
I_{7,1} &= \gamma \frac{M_8}{8!} - 7\gamma' \frac{M_9}{9!} + 28\gamma'' \frac{M_{10}}{10!} - \dots \\
I_{8,1} &= \gamma \frac{M_9}{9!} - 8\gamma' \frac{M_{10}}{10!} + 36\gamma'' \frac{M_{11}}{11!} - \dots \\
I_{9,1} &= \gamma \frac{M_{10}}{10!} - 9\gamma' \frac{M_{11}}{11!} + 45\gamma'' \frac{M_{12}}{12!} - \dots \\
I_{10,1} &= \gamma \frac{M_{11}}{11!} - 10\gamma' \frac{M_{12}}{12!} + 55\gamma'' \frac{M_{13}}{13!}
\end{aligned}\right\} \qquad (43.63)$$

The only redundant unknown in this particular problem is the support moment $\overline{M}_A$; it is determined from the boundary condition that at the right support the deflection is zero. The n-th moment at the right support is

$$\frac{M_n}{n!}=\frac{n-1}{2}\cdot\frac{ql^{n+1}}{(n+1)!}-\overline{M}_A\frac{l^{n-1}}{(n-1)!} \tag{43.64}$$

When $v=0$ and $x=l$ from (43.23) considering expressions (43.60)-(43.64) we obtain

$$\begin{aligned}&\gamma_0\left(\frac{M_3}{3!}+3\frac{8\alpha}{l^2}\cdot\frac{M_5}{5!}\right)-S\left[\gamma_0^2\frac{M_5}{5!}+13\gamma_0^2\frac{8\alpha}{l^2}\cdot\frac{M_7}{7!}\right]+\\&+S^2\left[\gamma_0^2\left(\gamma_0\frac{M_7}{7!}+21\gamma_0\frac{8\alpha}{l^2}\cdot\frac{M_9}{9!}\right)+\right.\\&+13\gamma_0^2\frac{4\alpha}{l^2}\left(\gamma_0\frac{M_9}{9!}+36\gamma_0\frac{8\alpha}{l^2}\cdot\frac{M_{11}}{11!}\right)+\\&\left.+63\gamma_0^2\left(\frac{8\alpha}{l^2}\right)^2\left(\gamma_0\frac{M_{11}}{11!}+55\gamma_0\frac{8\alpha}{l^2}\cdot\frac{M_{13}}{13!}\right)\right]-\ldots=0\end{aligned} \tag{43.65}$$

Taking the second approximation which gives an overestimated value of the support moment, we arrive at the equation

$$\gamma_0\frac{M_3}{3!}+\left(\frac{24\alpha}{l^2}\gamma_0-S\gamma_0^2\right)\frac{M_5}{5!}-\frac{104\alpha}{l^2}S\gamma_0^2\frac{M_7}{7!}=0 \tag{43.66}$$

from which, after substituting the values of the higher moments calculated according to formula (43.64) and rearranging the terms, the unknown quantity is determined

$$M_A=\frac{\left(1+\frac{8}{5}\alpha\right)-S\gamma_0l^2\left(\frac{1}{15}+\frac{13}{10}\alpha\right)}{(1+2\alpha)-S\gamma_0l^2\left(\frac{1}{12}+\frac{13}{45}\alpha\right)}\cdot\frac{ql^2}{12} \tag{43.67}$$

44. HIGHER MOMENTS OF VECTOR QUANTITIES IN SPACE

1. Definition of Higher Moments in Space

In examining higher moments in space, arbitrary vector quantities are considered and rules are established to determine moments of any order of these quantities given in space.

The moment of the n-th order of a vector P is represented by the product Ph^n, where h is the arm of the vector P with respect to the given centre of the moments.

2. Higher Moments of Vector, Arbitrarily Defined in Plane

Examining the two-dimensional problem successively in three mutually perpendicular planes of a left-handed coordinate system and compiling expressions for the first moments of the three components X, Y, Z of any vector quantity, for example, of a force, which is located arbitrarily in space and represented by vector P, in all three cases, we place the observer in identical conditions for observing the apparent rotation caused by the moment with respect to the coordinate axes, while the origin is located at the centre of moments. According to vector calculus, the sign of a linear moment is assumed positive when the rotation caused by the moment takes place in the direction of the shortest way of the axis x rotating about the axis z to coincide with the axis y.

If the point of application of an arbitrary vector is given by coordinates x, y, z, then the components of the linear moment of the vector P with respect to the moment centre O along the axes x, y, z (Fig. 44.1) are expressed by formulas

$$\left.\begin{aligned} M_{1x} &= yZ - zY \\ M_{1y} &= zX - xZ \\ M_{1z} &= xY - yX \end{aligned}\right\} \qquad (44.1)$$

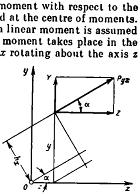

Fig. 44.1

They represent the moments of the vector P with respect to axes x, y and z, respectively.

Let us examine how these moments are formed, as well as the moments of higher order, with respect to axis x.

Let h_x be the arm of the vector P_{yz} with respect to point O; then

$$M_{1x} = P_{yz}h_x = P_{yz}\,(y\cos\alpha - z\sin\alpha)$$

However,

$$\cos\alpha = \frac{Z}{P_{yz}} \quad \text{and} \quad \sin\alpha = \frac{Y}{P_{yz}}$$

Designating $M_1(Z) = yZ$ and $M_1(Y) = zY$, we obtain

$$M_{1x} = M_1(Z) - M_1(Y)$$

In all these cases subscript 1 means that the arm of the force is taken in the first power. Similarly, for the second moments designation M_2 will be used, etc.

The second moment of the same component P_{yz} of the vector P with respect to the moment centre O is given by

$$\begin{aligned} M_{2x} &= P_{yz}h_x^2 = P_{yz}\,(y\cos\alpha - z\sin\alpha)^2 = P_{yz}y^2\cos^2\alpha - \\ &\quad - 2P_{yz}yz\cos\alpha\sin\alpha + P_{yz}z^2\sin^2\alpha = P_{yz}y^2\frac{Z^2}{P_{yz}^2} - 2P_{yz}yz\frac{Y}{P_{yz}} \times \\ &\quad \times \frac{Z}{P_{yz}} + P_{yz}z^2\frac{Y^2}{P_{yz}^2} = \frac{1}{P_{yz}}\,[M_2(Z)\,Z - 2M_1(Z)\,M_1(Y) + M_2(Y)\,Y] \end{aligned}$$

where

$$M_2(Z) = Zy^2; \quad M_2(Y) = Yz^2$$

$$M_1(Z) = Zy; \quad M_1(Y) = Yz; \quad P_{yz} = \sqrt{Z^2 + Y^2}$$

Similarly, for the third moment, we can write

$$M_{3x} = P_{yz} h_x^2 = \frac{1}{P_{yz}^2} [M_3(Z) Z^2 - 3M_2(Z) ZM_1(Y) + \\ + 3M_1(Z) M_2(Y) Y - M_3(Y) Y^2], \text{ etc.}$$

Formulas for moments of various orders may be presented in the form of moment algorithms

$$\left.\begin{array}{l} M_{1x}(P_{yz}) P_{yz}^0 = M_1(Z) Z^0 - M_1(Y) Y^0 \\ M_{2x}(P_{yz}) P_{yz} = M_2(Z) Z - 2M_1(Z) Z^0 M_1(Y) Y^0 + M_2(Y) Y \\ M_{3x}(P_{yz}) P_{yz}^2 = M_3(Z) Z^2 - 3M_2(Z) ZM_1(Y) Y^0 + \\ + 3M_1(Z) Z^0 M_2(Y) Y - M_3(Y) Y^2 \\ M_{nx}(P_{yz}) P_{yz}^{n-1} = M_n(Z) Z^{n-1} - nM_{n-1}(Z) Z^{n-2} M_1(Y) Y^0 + \\ + \frac{n(n-1)}{1 \cdot 2} M_{n-2}(Z) Z^{n-3} M_2(Y) Y - \\ - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} M_{n-3}(Z) Z^{n-4} M_3(Y) Y^2 + \\ + \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4} M_{n-4}(Z) Z^{n-5} M_4(Y) Y^3 - \ldots \\ \ldots + (-1) M_n(Y) Y^{n-1} \end{array}\right\} \quad (44.2)$$

The first equation of this system represents Varignon's theorem

$$M_1(P_{yz}) = M_1(Z) Z^0 - M_1(Y) Y^0$$

which is obtained as a particular case of the moment M_n when $n = 1$.

According to formulas (44.2), the coefficients of the polynomial terms in all the moment algorithms are the usual binomial coefficients determined by Pascal's triangle (see Table 18.1).

All the moment algorithms in (44.2) are products of two factors, one of which is a moment and the other is a vector quantity, from which the moment is derived. The structure of these products is distinctive—it corresponds to the structure of static invariants known in theoretical mechanics; they are expressed in the form $LX + MY + NZ = \text{const}$ and also present moments multiplied by forces.

Algorithms for moments M_y and M_z with respect to two other coordinate axes y and z are compiled similar to formulas (44.2).

If all moments in both parts of equations (44.2) are divided by a factorial denominator $n!$, the equalities remain valid and formulas

(44.2) become free from the binomial coefficients

$$\frac{M_{nx}}{n!} P_{yz}^{n-1} = \frac{M_n(Z)}{n!} Z^{n-1} - \frac{M_{n-1}(Z)}{(n-1)!} Z^{n-2} M_1(Y) Y^0 +$$
$$+ \frac{M_{n-2}(Z)}{(n-2)!} Z^{n-3} \frac{M_2(Y)}{2} Y - \dots (-1)^n \frac{M_n(Y)}{n!} Y^{n-1} \qquad (44.3)$$

The application of formulas (44.2) and (44.3) is not limited by the case of one vector quantity. If, for example, the components S and T of the vector P (Fig. 44.2) are not parallel to the coordinate axes z and y, then the problem of determining the moments of this vector may be reduced to the previous problem by employing components Y_1, Z_1, Y_2 and Z_2 of the given components S and T. In this case formulas (44.2) and (44.3) remain the same

$$Z = Z_1 + Z_2 \text{ and } Y = Y_1 + Y_2$$

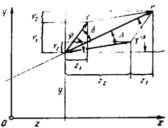

Fig. 44.2

In the case of more than two intersecting vectors, i.e., a pencil of m vectors with a common origin, the components Z_i and Y_i of each vector P_i may be determined likewise, and in formulas (44.2) and (44.3)

$$Z = \sum_{i=1}^{i=m} Z_i \quad \text{and} \quad Y = \sum_{i=1}^{i=m} Y_i$$

3. Sign Convention for Higher Moments of Arbitrary Vector in Plane

Suppose a vector P is given in the plane yz (Fig. 44.1) by its components Z and Y, the coordinates of its origin being z and y.

According to Fig. 44.1, the signs of Z, Y, z and y are positive, and the signs "plus" and "minus" in front of the terms yZ and zY in formula (44.1) account for different directions of rotation caused by components Z and Y with respect to the moment centre O. Corresponding sign conventions for moments of any order may be established on the basis of this particular sign convention for linear moments of a vector with respect to an arbitrary point in its plane. These rules make it possible to determine the sign of a moment M_n for any n directly from the position of the vector in the plane.

If all the coordinates of the vector from which the moments are to be determined are included in (44.2) and (44.3) with their signs, then the quantities M_1P^0, M_2P, M_3P^2, etc. have signs which correspond to the direction of rotation in the chosen left-handed coordinate system.

In the formation of a moment of a vector with respect to a point, the sign "plus" or "minus" refers both to the vector and to the arm of its action. The vector of the arm in the plane yz is given by

$$h = y \cos \alpha - z \sin \alpha \tag{44.4}$$

where the coordinates of the origin of the vector (z, y) are included with their signs. Angle α is the angle between the coordinate axis z and the vector. Thus, the vector P (Fig. 44.1) is determined by its direction (angle α between the vector and axis z) and by its sign (position with respect to the axis y); the vector is considered positive if its projection on the y axis is positive. If the vector is parallel to the z axis, then its projection on the y axis is zero and the sign of this vector corresponds to the direction of the z axis.

A similar sign convention applies to the arm h if it is always considered towards the moment centre: the arm is positive if its projection on the z axis is positive.

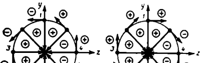

Fig. 44.3

Let us check this rule. For example, the abscissa z of point *1* (Fig. 44.3) is zero; therefore according to formula (44.4), it follows that $h = y \cos \alpha$, where y and $\cos \alpha$ are positive; hence, the arm h is positive. Similarly, the arm h of the lowest point *2* is negative. For the extreme left point *3* the ordinate $y = 0$, hence $h = -z \sin \alpha$, where z is negative and $\sin \alpha$ is positive; therefore, h is positive, etc. Fig. 44.3 presents the signs of the vectors and their arms for different positions of the vector; the sign of the arm, when determined according to formula (44.4), is the same as the sign determined according to the above described rule of projections. If the arm h coincides with the y axis, then its projection on the z axis is zero and the y axis should be used instead of the z axis to determine the sign of the arm: the arm is considered positive if it points towards the positive direction of the y axis, and negative if it points towards the negative direction of the y axis.

These sign conventions apply to planes zx and xy as well as after cyclic permutations of the subscripts of the coordinate axes.

Formulas (44.2) and (44.3) are alternating polynomials in which all the coordinate data are included with their signs. If the signs of moments which are included in separate terms of these polynomials and relate to the components of the P vector are determined according to the presented projection rule, then all the polynomials should be assumed non-alternating.

If the projection rule is followed for both the vector and its arm, then linear moments are positive when rotating clockwise, regardless of the position of the vector.

According to the vector calculus, the moment of the vector P (Fig. 44.4) with respect to the moment centre O is determined by the vector product $[P, r]$ which is equal to the area of the parallelogram with vectors P and r as its sides.

Thus, $M_1(P) = [P, r] = Ph$. The moment being a vector product is represented by a vector which is perpendicular to both vector factors. The direction of the vector of the moment must be chosen so that the shortest way for the vector product $[P, r]$ to rotate from P to r coincides with that for the x axis to the y axis. Thus, according to Fig. 44.4, the vector product $[P, r]$ is positive, while $[r, P]$ is negative, because the commutative property does not apply to the vector product.

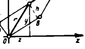

Fig. 44.4

A product of the form Ph^n cannot be considered a vector product, since the arm h is not a radius-vector of the origin of the vector P. However, products of this form may be conventionally assumed to be vector products, if the vector coordinates of the applied vector are altered and the arm h of the moment formed by this vector is taken as a vector factor of the vector product.

Let us examine the conditions under which the vector calculus may be used in the theory of higher moments.

The vector product $[P, h]$ for a linear moment does not contradict the definitions of the vector calculus, because at least in one particular case — when the radius-vector is perpendicular to its vector which forms the moment (point C is the point of application of the vector)—we obtain

$$[P, r] = [P, h]$$

Let us check all the other conditions to be satisfied by the arm h when it is considered a vector factor.

1. The arm h must be represented by a polar vector, as well as the radius-vector r. This condition is satisfied for higher moments since the arm is determined by expression $h = y \cos \alpha - x \sin \alpha$, which means that it has a definite sign and may be represented by a vector.

2. The arm h, as well as the radius-vector r, must have a common origin with the given vector forming the moment. This follows from the definition of the vector product, which is equal to the area of a parallelogram constructed by the vector factors; the sign of this area depends on the order of the factors. According to Fig. 44.4, the arm $h = \overline{AB}$; therefore, vector $AB = h$ may be considered as

being applied at the point of application of the given vector P, i.e., at point A.

3. The vector $\boldsymbol{h}$ must be located with respect to vector P in such a way, that the sign convention of the vector calculus can be applied to the vector product $[P, h]$. It holds for the linear moment of the vector P about the moment centre O, because the rotation of P to $\boldsymbol{h}$ takes place in the same direction as the rotation of P to the radius-vector r, in the plane of the coordinate axes, regardless of the position of the vector or of its origin A.

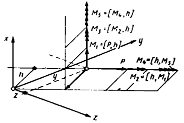

Fig. 44.5

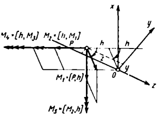

Fig. 44.6

4. The conditional moment vector products of the form $[P, h^n]$ must be represented by a vector which is perpendicular to the vector factors, thus allowing graphical interpretation. In Fig. 44.4 both the moment-forming vector and the arm vector are positive; therefore, the n-th moment will be positive for any value of n.

Let us represent higher moments by vectors (Fig. 44.5), considering the non-commutative property of vector products.

The vectors representing moments and vector factors in Fig. 44.5 differ from each other: the first moment is represented by a single arrow, the second moment—by a double arrow, etc.

The vectors of odd moments are normal to the plane yz of their action; they are directed along the positive half of the x axis. The vectors of even moments are located in the yz plane and, when they are positive, directed along the fundamental vector P.

If vector P is negative and h is positive, then all the moments are negative, as shown in Fig. 44.6. According to the rules stated above, the odd moments (in this particular case) are directed along the negative half of the x axis, while the even moments coincide with vector P, which is negative. The rectangles corresponding to the vector products are drawn in thin lines in Figs. 44.5-44.7.

If both the P vector and its arm h are negative, then the moment vectors are arranged as shown in Fig. 44.7. In this case the odd moments are positive and the even moments are negative.

Thus, higher moments may be also represented by vectors alternating the order of vector factors in even and odd moments. The vectors of moments are always perpendicular to the vector factors; vectors of odd moments are perpendicular to the plane of the moments' action and directed along the positive half of the coordinate axis, which is perpendicular to the plane of the moments' action if the moment is positive; the direction of a negative odd moment is opposite. Vectors of even moments lie in the plane of their action and always coincide in direction with the vector, which forms these moments; the sign of an even moment corresponds to the sign of the vector.

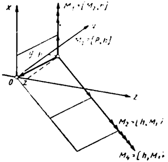

Fig. 44.7

The above presentation leads to the following conclusions.

Vector products $[P, r]$ and $[P, h]$ in the formulas for higher moments represent equivalent linear moments $M_1 = [P, r] = [P, h]$.

In order to determine the second moment $M_2 = Ph^2 = (Ph)\,h = hM_1$, we must find the vector product of the arm h and the first moment

$$M_2 = [h, M_1] = [h, [P, h]]$$

Therefore, the second moment may be interpreted as a double vector product.

Similarly, the third moment

$$M_3 = [M_3, h] = \{[h, [P, h]], h\}$$

is a conventional triple vector product, etc.

Any moment of higher order may be conventionally considered a multiple vector product of the primary vector factors P and h which are mutually perpendicular. Odd order moments are vector products of the initial vector and its arm, while the even order moments are vector products of the arm and the vector. This alternation of vector factors with the increase in the order of the moment accounts for the non-commutativeness of the vector product, which corresponds to the general sign convention. Vectors of odd moments coincide with a perpendicular to the plane of the moment and pass through the origin of the moment-forming vector; vectors of even moments coincide with the moment-forming vector.

The property of vector double products

$$[a, [b, c]] = b\,(a, c) - c\,(a, b)$$

applies to moments of higher order.

For example, in the case of the second moment

$$[M_2] = [h, M_1] = [h, [P, h]]$$

we obtain

$$M_2 = P\,(h, h) - h\,(h, P) = Ph^2$$

because the scalar products are

$$(h, h) = h^2 \quad \text{and} \quad (h, P) = 0$$

Free vectors of the first order may be added and subtracted according to vector algebra regardless of their position in space. As for the higher moments, both odd and even, the rules governing their addition are specific, as it follows from formula (44.2), although the odd moments coincide with the vector of the first moment, while the even moments coincide with the initial vector itself. Formula (44.2) can be rewritten for the second moment according to its definition

$$\begin{aligned} M_2(P_{yz}) &= P_{yz}h_x^2 = P_{yz}(y\cos\alpha - z\sin\alpha)^2 = \\ &= P_{yz}y^2\cos^2\alpha - 2P_{yz}\cos\alpha\sin\alpha + P_{yz}z^2\sin^2\alpha = \\ &= P_{yz}y^2\frac{Z^2}{P_{yz}^2} - 2P_{yz}yz\cos\alpha\sin\alpha + P_{yz}z^2\frac{Y^2}{P_{yz}^2} = \\ &= (Zy^2)\frac{Z}{P_{yz}} - (P_{yz}yz)\sin 2\alpha + (Yz^2)\frac{Y}{P_{yz}} = \\ &= M_2(Z)\cos\alpha - M_{yz}\sin 2\alpha - M_2(Y)\sin\alpha \end{aligned}$$

where $M_{yz} = P_{yz}yz$ represents a biaxial moment of the vector P_{yz}.

The first and third terms of this expression are the components of second moments of the Z and Y components of the vector P_{yz}; they pass through the origin of this vector. Consequently, the second term of this expression $M_{yz}\sin 2\alpha$ represents a vector which also passes through the origin of P_{yz}, because all the terms of the expression must be homogeneous. If this second term is to be interpreted as a corresponding projection of a biaxial moment, then the vector of such a biaxial moment M_{yz} may be constructed according to the following rule: it must form an angle (90° − 2α) with the vector P_{yz}.

The algorithm of the third moments also agrees with this conception. The first and third terms of the expression for the third moment $M_3(P_{yz})$, as shown by formula (44.2), may be interpreted as a result of two successive projections (direct and reverse) of vectors $M_3(Z)$ and $M_3(Y)$. These vectors are perpendicular to the yz plane and so are their second projections, because the projection only reduces the length of the vectors. Therefore, the other terms of the polynomial for the third moment, according to (44.2), also have the same origin with the vector $M_3 (P_{yz})$ due to homogeneity of terms.

4. Relationship Between Higher Moments of Vector Defined in Plane with Respect to Two Points of the Same Plane

If any applied vector P (Fig. 44.8) is given in the plane of coordinate axes z and y, and it is translated to another point C_1 in the same plane, which is equivalent to a vector couple with an arm h and a vector P applied at point C_1; therefore, the moment of the vector P about any other moment centre C_2 in the same plane is equal to the sum of the moment (Ph) and the moment (Pc) of the vector P about the point C_2.

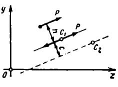

Fig. 44.8

Let us examine a similar relationship between higher moments about two moment centres C_1 and C_2 arbitrarily located in a plane.

Let $M_1, M_2, M_3, \ldots$ represent the moments of the vector P about the point C_1 and $M_{11}, M_{22}, M_{33} \ldots$ represent the moments of the same vector about the point C_2. Then

$$M_{11} = P(h+c) = Ph + Pc = M_1 + Pc$$

$$M_{22} = P(h+c)^2 = Ph^2 + 2Phc + Pc^2 = M_2 + 2M_1c + Pc^2$$

$$M_{33} = P(h+c)^3 = Ph^3 + 3Ph^2c + 3Phc^2 + Pc^3 =$$
$$= M_3 + 3M_2c + 3M_1c^2 + Pc^2, \text{ etc.}$$

Relationships between higher moments may be based on the structure of these algorithms revealed in the previous expressions

$$M_{nn} = M_nc^0 + nM_{n-1}c + \frac{n(n-1)}{1\cdot 2} M_{n-2}c^2 + \\ + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} M_{n-3}c^3 + \ldots + \\ + \frac{n(n-1)}{1\cdot 2} M_2c^{n-2} + nM_1c^{n-1} + M_0c^n \qquad (44.5)$$

Coefficients of the terms in formulas (44.5) are binomial coefficients, corresponding to Pascal's triangle.

If the second moment centre C_2 is nearer to the vector P than point C_1, then c in (44.5) is replaced by $(-c)$ and all the polynomials become alternating.

A polynomial obtained for a moment of any order is a series of moment products of the vector P about one of the points C_1 or C_2

and the intercentre distance c, with binomial coefficients. These products are constructed so that the order of the moments decreases from n in the first term to zero in the last term, while the power of the intercentre distance increases from zero to n. The sum of the order of the moment and the power of c is constant and equal to n for all the terms of the polynomial. The number of terms in the polynomial is $(n+1)$. All the terms of a polynomial have the same sign when it expresses the moment about the farther moment centre C_2 in terms of the moment about the nearer moment centre C_1; the inverse relationship is expressed by an alternating polynomial.

The relationship between linear moments about two moment centres C_1 and C_2, which is known in theoretical mechanics, may be derived from the general formula for M_{nn} as a particular case when $n=1$.

Factorial denominators may be introduced in formulas (44.5), thus eliminating the binomial coefficients of all the terms. Then

$$\frac{M_{nn}}{n!}=\frac{M_n}{n!}c^0+\frac{M_{n-1}}{(n-1)!}c-\frac{M_{n-2}}{(n-2)!}\frac{c^2}{2}+\dots$$
$$\dots+\frac{M_2}{2}\cdot\frac{c^{n-2}}{(n-2)!}+M_1\frac{c^{n-1}}{(n-1)!}+M_0\frac{c^n}{n!} \tag{44.6}$$

If the higher moments of moment vectors are nominated as bimoments, then formulas (44.5) may be considered as a bimoment series of various orders with an arm equal to the intercentre distance.

By introducing the designation $\mathfrak{M}$ for bimoments, formulas (44.5) become

$$M_{nn}=\mathfrak{M}_{n_0}+n\mathfrak{M}_{n-1,\,1}+\frac{n(n-1)}{1\cdot 2}\mathfrak{M}_{n-2,\,2}+$$
$$+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\mathfrak{M}_{n-3,\,3}+\dots$$
$$\dots+\frac{n(n-1)}{1\cdot 2}\mathfrak{M}_{2,\,n-2}+n\mathfrak{M}_{1,\,n-1}+\mathfrak{M}_{0,\,n} \tag{44.7}$$

The first part of every double subscript indicates the order of the moment which forms the bimoment, while the other part is the order of the bimoment itself.

Introduction of factorial denominators in (44.7) results in

$$\left.\begin{aligned}
&\frac{M_{11}}{1!}=\frac{\mathfrak{M}_{10}}{1!\;0!}+\frac{\mathfrak{M}_{01}}{0!\;1!};\;\frac{M_{22}}{2!}=\frac{\mathfrak{M}_{20}}{2!\;0!}-\frac{\mathfrak{M}_{11}}{1!\;1!}+\frac{\mathfrak{M}_{02}}{0!\;2!}\\
&\frac{M_{nn}}{n!}=\frac{\mathfrak{M}n_0}{n!\;0!}+\frac{\mathfrak{M}_{n-1,\,1}}{(n-1)!\;1!}+\frac{\mathfrak{M}_{n-2,\,2}}{(n-2)!\;2!}+\frac{\mathfrak{M}_{n-3,\,3}}{(n-3)!\;3!}+\dots\\
&\dots+\frac{\mathfrak{M}_{2,\,n-2}}{2!\;(n-2)!}+\frac{\mathfrak{M}_{1,\,n-1}}{1!\;(n-1)!}+\frac{\mathfrak{M}_{0,\,n}}{0!\;n!}
\end{aligned}\right\} \tag{44.8}$$

The signs of bimoments are determined according to the following rule. Odd bimoments are positive when the bimoment is determined about the farthest of the two moment centres, and it is negative when determined about the nearest moment centre; even bimoments are always positive.

5. Higher Moments of Arbitrary Force (General Case)

Let us examine the higher moments of a vector P located arbitrarily in space. Let X, Y, Z represent the components of P parallel to the axes x, y, z, respectively, and x_0, y_0, z_0—the coordinates of the origin of vector P. Let the moment centre coincide with the origin O.

The following vector expression for the moment of the force about any point in space can serve as a basis for the initial algorithms

$$M_1 = \sqrt{M_{1x}^2 + M_{1y}^2 + M_{1z}^2} \tag{44.9}$$

In this expression, according to (44.9),

$$M_{1x} = M_1(Z)\,Z^0 - M_1(Y)\,Y^0$$
$$M_{1y} = M_1(X)\,X^0 - M_1(Z)\,Z^0$$
$$M_{1z} = M_1(Y)\,Y^0 - M_1(X)\,X^0$$

Since

$$M_{1x}^2 = P_{yz}^2 h_x^2 = P_{yz} h_x^2 P_{yz} = M_{2x} M_{0x}$$
$$M_{1y}^2 = P_{yz}^2 h_y^2 = P_{zx} h_y^2 P_{zx} = M_{2y} M_{0y}$$
$$M_{1z}^2 = P_{xy}^2 h_z^2 = P_{xy} h_z^2 P_{xy} = M_{2z} M_{0z}$$

we obtain

$$M_1^2 = M_{2x} M_{0x} + M_{2y} M_{0y} + M_{2z} M_{0z} \tag{44.10}$$

which is the main expression for the moment algorithm in space; any higher moment of a given vector

$$P = \sqrt{P_{yz}^2 + P_{zx}^2 + P_{xy}^2}$$

can be expressed in terms of this algorithm, as will be shown below.

Denoting the main expression for the moment algorithm through M_s we find according to (44.2)

$$\begin{aligned} M_s = {} & M_{2x} M_{0x} + M_{2y} M_{0y} + M_{2z} M_{0z} = M_2(Z)\,Z - \\ & - 2M_1(Z)\,Z^0 M_1(Y)\,Y^0 + M_2(Y)\,Y + M_2(X)\,X - \\ & - 2M_1(X)\,X^0 M_1(Z)]Z^0 + M_2(Z)\,Z + \\ & + M_2(Y)\,Y - 2M_1(Y)\,Y^0 M_1(X)\,X^0 + M_2(X)\,X \end{aligned} \tag{44.11}$$

Thus, the main moment algorithm is determined by an algebraic sum of the second moments in three mutually perpendicular coordinate planes; each second moment is multiplied by a component of the moment-forming vector acting in the same plane.

To determine the higher moments of vector P in space, the following expressions are used

$$\left.\begin{aligned} M_{\mathrm{I}}^{2} &= P^{2}h^{2} = Ph^{2}P = M_{\mathrm{II}}P \\ M_{\mathrm{I}}^{3} &= P^{3}h^{3} = Ph^{3}P^{2} = M_{\mathrm{III}}P^{2} \\ &\cdots\cdots\cdots\cdots\cdots \\ M_{\mathrm{I}}^{n} &= P^{n}h^{n} = Ph^{n}P^{n-1} = M_{n}P^{n-1} \end{aligned}\right\} \tag{44.12}$$

According to formulas (44.10) and (44.11), $M_{\mathrm{I}}^{2} = M_{s}$, therefore, according to (44.12),

$$M_{\mathrm{II}}P = M_{s} \tag{44.13}$$

and

$$M_{\mathrm{III}}^{2}P^{4} = M_{s}^{3} \tag{44.14}$$

Similarly,

$$M_{\mathrm{IV}}P^{3} = M_{\mathrm{I}}^{2}M_{\mathrm{I}}^{2} = M_{s}^{2} \quad \text{and} \quad M_{\mathrm{V}}P^{4} = M_{s}^{2}\sqrt{M_{s}}$$

hence,

$$M_{\mathrm{V}}^{2}P^{8} = M_{s}^{5}, \text{ etc.}$$

Comparing the derived formulas we obtain the following expressions for even and odd moments

$$\left.\begin{aligned} M_{\mathrm{I}}^{2}P^{0} &= M_{s} & M_{\mathrm{II}}P &= M_{s} \\ M_{\mathrm{III}}^{2}P^{4} &= M_{s}^{3} & M_{\mathrm{IV}}P^{3} &= M_{s}^{2} \\ M_{\mathrm{V}}^{2}P^{8} &= M_{s}^{5} & M_{\mathrm{VI}}P^{5} &= M_{s}^{3} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{2n-1}^{2}P^{4(n-1)} &= M_{s}^{2n-1} & M_{2n}P^{2n-1} &= M_{s}^{n} \end{aligned}\right\} \tag{A}$$

Thus, formulas (A) reduce the problem of determining higher moments of an arbitrary vector P to calculations of various powers of the main moment algorithm M_s. Squared odd moments are multiplied in these formulas by the vector quantity P which is raised to a power equaling the sum of two orders: of the moment determined and of the preceding odd moment. For example, the exponent of the factor P for M_{VII} equals the sum of 5 and 7, i.e., 12, etc. The exponent of the factor P in even moments is a unit less than the order of the moment determined. Finally, there is a general rule which is true for both even and odd moments: the exponent of the main algorithm M_s required for calculation of higher moments, is half the sum of the exponents of M and P. For example, to determine M_{VIII} it is necessary to calculate M_s in the fourth power, because the sum of exponents of M_{VIII} and P^7 is 8, etc. For odd moments the exponent of M_s equals the order of the moment determined; it is half this order for even moments.

All these relationships are taken into account in the general formulas for moments of orders $2n$ and $2n - 1$, where n is a positive integer.

Substitution of the expression for M_s from (44.11) into (A) results in the final of moment algorithms after the trinomial $M_{2x}M_{0x} + M_{2y}M_{0y} + M_{2z}M_{0z}$ was considered as the sum of the binomial $M_{2x}M_{0x} + M_{2y}M_{0y}$ and monomial $M_{2z}M_{0z}$.

For odd moments

$$\left.\begin{aligned}
&M_{\text{I}}^2P^0 = M_{2x}M_{0x} + M_{2y}M_{0y} + M_{2z}M_{0z}\\
&M_{\text{III}}^2P^4 = M_{2x}^3M_{0x}^3 + 3M_{2x}^2M_{0x}^2M_{2y}M_{0y} +\\
&+ 3M_{2x}M_{0x}M_{2y}^2M_{0y}^2 + M_{2y}^3M_{0y}^3 +\\
&+ 3\,[M_{2x}^2M_{0x}^2 + 2M_{2x}M_{0x}M_{2y}M_{0y} + M_{2y}^2M_{0y}^2]\,M_{2z}M_{0z} +\\
&+ 3\,[M_{2x}M_{0x} + M_{2y}M_{0y}]\,M_{2z}^2M_{0z}^2 + M_{2z}^3M_{0z}^3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&M_{2n-1}^2P^{4(n-1)} = M_{2x}^{2n-1}M_{0x}^{2n-1} + (2n-1)\times\\
&\times M_{2x}^{2n-2}M_{0x}^{2n-2}M_{2y}M_{0y} + \frac{(2n-1)(2n-2)}{1\cdot 2}\times\\
&\times M_{2x}^{2n-3}M_{0x}^{2n-3}M_{2y}^2M_{0y}^2 + \ldots + (2n-1)\Big[M_{2x}^{2n-2}M_{0x}^{2n-2} +\\
&+ (2n-2)\,M_{2x}^{2n-3}M_{0x}^{2n-3}M_{2y}M_{0y} + \frac{(2n-2)(2n-3)}{1\cdot 2}\times\\
&\times M_{2x}^{2n-4}M_{0x}^{2n-4}M_{2y}^2M_{0y}^2 + \ldots\Big]M_{2z}M_{0z} + \frac{(2n-1)(2n-2)}{1\cdot 2}\times\\
&\times\Big[M_{2x}^{2n-3}M_{0x}^{2n-3} + (2n-3)\,M_{2x}^{2n-4}M_{0x}^{2n-4}M_{2y}M_{0y} +\\
&+ \frac{(2n-3)(2n-4)}{1\cdot 2}M_{2x}^{2n-5}M_{0x}^{2n-5}M_{2y}^2M_{0y}^2 + \ldots\Big]\times\\
&\times M_{2z}^2M_{0z}^2 + \ldots + M_{2z}^{2n-1}M_{0z}^{2n-1}
\end{aligned}\right\}\quad(44.15)$$

For even moments

$$\left.\begin{aligned}
&M_{\mathrm{II}}P^{1} = M_{2x}M_{0x} + M_{2y}M_{0y} + M_{2z}M_{0z}\\
&\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&M_{2n}P^{2n-1} = M_{2x}^{n}M_{0x}^{n} + nM_{2x}^{n-1}M_{0x}^{n-1}M_{2y}M_{0y} + \frac{n(n-1)}{1\cdot 2}\times\\
&\times M_{2x}^{n-2}M_{0x}^{n-2}M_{2y}^{2}M_{0y}^{2} + \ldots + n\Big[M_{2x}^{n-1}M_{0x}^{n-1} + (n-1)\times\\
&\times M_{2x}^{n-2}M_{0x}^{n-2}M_{2y}M_{0y} + \frac{(n-1)(n-2)}{1\cdot 2}\times\\
&\times M_{2x}^{n-3}M_{0x}^{n-3}M_{2y}^{2}M_{0y}^{2} + \ldots\Big]M_{2z}M_{0z} + \frac{n(n-1)}{1\cdot 2}\times\\
&\times\Big[M_{2x}^{n-2}M_{0x}^{n-2} + (n-2)M_{2x}^{n-3}M_{0x}^{n-3}M_{2y}M_{0y} +\\
&+\frac{(n-2)(n-3)}{1\cdot 2}M_{2x}^{n-4}M_{0x}^{n-4}M_{2y}^{2}M_{0y}^{2} + \ldots\Big]M_{2z}^{2}M_{0z}^{2} +\\
&+\ldots + M_{2z}^{n}M_{0z}^{n}
\end{aligned}\right\}\quad (44.16)$$

Thus, the moment algorithms of the form given in formulas (44.2), (44.3) and (44.15), (44.16) may be used in all cases of applied theoretical investigations and in research work, when vector quantities in space are multiplied by another vector quantity raised to a power higher than unity. These formulas give solutions in terms of higher moments, the moment algorithms derived previously reduce calculations for high powers of conventional vector products to simple operations with the initial vectors which are given by their coordinates.

45. RIGIDITY OF BEAMS: GENERAL CASE OF NON-LINEAR STRESS-STRAIN RELATIONSHIP

The problems involved in determining rigidity of beams on the basis of a non-linear stress-strain relationship was elucidated by a great number of scientists in many countries. Various calculation procedures were proposed, both analytical and grapho-analytical. All the scientists accepted Bernoulli's hypothesis of plane sections and assumed that the stress-strain relationships obtained for uniaxial tension and compression can be applied to other phenomena, such as bending.

Some authors examined the σ-ε diagram directly, without analytical approximations. N. Bezookhov, as mentioned previously, suggested to represent the $\sigma = f(\varepsilon)$ curve by a broken line. Other scientists approximated the σ-ε curve by various analytical relationships.

In the present analysis we shall assume the following relationship

$$\sigma = K\varepsilon^m \qquad (45.1)$$

which is usually attributed to Bach and Shulle, but was indeed first suggested by Bulfinger and published in the Transactions of the Russian Academy of Sciences in 1729, as pointed out by N. Mitinsky.

This formula was also used by other scientists; for example, L. Kachanov, N. Malinin and I. Morin applied it to the steady-state creep case, when the coefficient K was assumed to vary with time.

Yu. Rabotnov suggested the following relationship

$$\sigma = \varphi(\varepsilon)\,\psi(t,\ T)$$

where t and T are time and temperature, respectively.

This relation meant that Yu. Rabotnov assumed the creep curves to be similar for different stress levels. The same conclusion follows from formula (45.1), so the design method presented below may be applied to analyse bent beams in steady-state creep.

All calculations will be carried out on the basis of formula (45.1) assuming that the material behaves identically under tension and compression.

In deflection of a beam, the cross section of which has one axis of symmetry (the plane of forces passes through this axis), the displacement of the neutral axis from the centre of gravity takes place. In some cases, when the cross section has two axes of symmetry, this displacement is so small that it can be neglected. However, without a preliminary estimate of the possible error, displacement should not be disregarded.

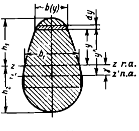

Fig. 45.1

The position of the neutral axis (Fig. 45.1) is determined from equation

$$\int^{F} \sigma\, dF = 0 \qquad (45.2)$$

In flexure the strain is determined by

$$\varepsilon = \frac{y^x}{\rho}$$

where y^x = distance from the neutral axis to the fibre under consideration

ρ = radius of curvature of the flexural curve in the considered section

The following is derived from formulas (45.1) and (45.2)

$$I_{z1m}^{h_1+\gamma}-I_{z1m}^{h_2-\gamma}=0 \qquad (45.3)$$

where

$$I_{z1m}^{h_1+\gamma}=\int_0^{h_1+\gamma}|y^x|^m\,dF \quad \text{and} \quad I_{z1m}^{h_2-\gamma}=\int_0^{h_2-\gamma}|y^x|^m\,dF$$

are the moments of the m-th order of the parts of the cross section above and below the neutral axis, respectively, taken about the neutral axis.

For simple figures application of formula (45.3) does not meet with great difficulties, but for complicated figures, various approximate methods of integrating must be applied, because the position of the neutral axis is not determined. In this latter case another reference axis should be chosen parallel to the neutral axis. This axis will further be called the reference axis. Let us derive the transition formulas for these two axes.

Let m be a positive integer. Then

$$\int_0^{h_1+\gamma}(y^x)^m\,dF-\int_0^{h_2-\gamma}(y^x)^m\,dF=\int_{-\gamma}^{h_1}(y+\gamma)^m\,dF-\int_{\gamma}^{h_2}(y-\gamma)^m\,dF=$$

$$=(I_{zm}^{h_1}-I_{zm}^{h_2})+\frac{m}{1!}\gamma(I_{zm-1}^{h_1}+I_{zm-1}^{h_2})+$$

$$+\frac{m(m-1)}{2!}\gamma^2(I_{zm-2}^{h_1}-I_{zm-2}^{h_2})+\ldots+\gamma^m[I_{z0}^{h_1}-(-1)^m I_{z0}^{h_2}]+$$

$$+\sum\frac{C_i[(\gamma-L_i)^{m+1}-(\gamma-L_{i+1})^{m+1}][1+(-1)^m]}{m+1}=0 \qquad (45.4)$$

In this formula

$$I_{zm}^{h_1}=\int_0^{h_1}y^m\,dF \quad \text{and} \quad I_{zm}^{h_2}=\int_0^{h_2}y^m\,dF$$

are moments of the m-th order of the parts of the figure above and below the reference axis, respectively, taken about this axis (in most cases it is preferable to take the central axis for the reference axis).

The terms included in formula (45.4) under the sum sign should be calculated by dividing the part of the cross section between the neutral axis and the reference axis into rectangles of width C_i with distances L_i and L_{i+1} from the reference axis.

It can be noticed that γ reaches its maximum value when $m = 0$; in this particular case formula (45.4) may be written as

$$(I^{h_1}_{z0} - I^{h_2}_{z0}) + 2\sum C_i (L_{i+1} - L_i) = 0 \tag{45.5}$$

From this formula it follows that the maximum value of γ is determined from the condition that the difference of the areas of the figure above and below the reference axis plus the double area between the neutral and reference axes are equal to zero.

For fractional values of m formula (45.4) obtains the following form

$$(I^{h_1}_{zm} - I^{h_2}_{zm}) + \frac{m}{1!}\gamma (I^{h_1}_{zm-1} + I^{h_2}_{zm-1}) +$$

$$+ \frac{m(m-1)}{2!}\gamma^2 (I^{h_1}_{zm-2} - I^{h_2}_{zm-2}) + \ldots \approx 0 \tag{45.6}$$

Here is an example of determining $I^{h_1}_{zm}$ and $I^{h_2}_{zm}$ (Fig. 45.1)

$$I^{h_1}_{zm} = \frac{1}{m+1} [b_1 h_1^{m+1} + b_2 (h_2^{m+1} - h_1^{m+1}) + b_3 (h_3^{m+1} - h_2^{m+1})]$$

$$I^{h_2}_{zm} = \frac{1}{m+1} [B_1 H_1^{m+1} + B_2 (H_2^{m+1} - H_1^{m+1}) + B_3 (H_3^{m+1} - H_2^{m+1})]$$

Equation (45.3) shows that the displacement γ of the neutral axis does not depend on ρ, therefore, it does not depend on the bending moment M_x, but it does depend on the mechanical properties of the material, which are described by m, and on the geometrical characteristics of the cross section (i.e., the moments of various orders of the cross-sectional area). If the material has different characteristics under tension and compression, γ depends on the radius ρ as well.

The series in formula (45.6) converges rapidly, if the reference axis is properly chosen. A certain number of terms may be enough to give a value of γ sufficiently precise for practical use. The number of terms necessary for this purpose is established by evaluating the first of the rejected terms.

Let us examine the moment equality condition for external and internal forces

$$\int^{F} \sigma y^x \, dF = M_x$$

$$M_x = \frac{k}{\rho^m} [I^{h_1}_{z\,m+2} + I^{h_2}_{z\,m+1}] \tag{45.7}$$

Introducing the designation suggested by L. Kachanov

$$D^m = K (I^{h_1}_{z\,m+1} + I^{h_2}_{z\,m+1}) \tag{45.8}$$

and taking into account that $\frac{1}{\rho} \approx \frac{d^2V_x}{dx^2}$ for small deflections we arrive at

$$D\frac{d^2V_x}{dx^2} = M \tag{45.9}$$

Since the position of the neutral axis is determined by the previous formulas, there is no need to calculate $I_{z1m+1}^{h_1}$ and $I_{z1m+1}^{h_2}$ with respect to the reference axis.

Equation (45.9) obtains the following forms when we assume $m = 1$, $1/2$ and $1/3$, respectively,

$$D\frac{d^2V_x}{dx^2} = M_x; \quad D\frac{d^2V_x}{dx^2} = M_x^2; \quad D\frac{d^2V_x}{dx^2} = M_x^3 \tag{45.10}$$

In most cases, M_x has the form of an algebraic polynomial (except the case when the loading varies along the beam according to trigonometric, or any other law). Integration of equation (45.9) becomes difficult when the order of this moment is higher than unity, because, generally, $1/m$ is not an integer (for example, when continuous loads are applied to the beam).

A. Filin suggested to solve such problems for integer values of $1/m$, and to use approximations for fractional values of $1/m$. However, his method does not eliminate the difficulties in calculating the arbitrary constants of integration.

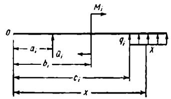

Fig. 45.2

These difficulties may be overcome by means of the theory of higher moments. Let us examine the main principles of this theory.

Consider an arbitrarily loaded system shown in Fig. 45.2 (the loading may be more complicated than the one in the figure). The moment M_n of the n-th order is determined by the expression

$$\frac{M_n}{n!} = \sum \frac{M_i\,(x-a_i)^{n-1}}{(n-1)!} + \sum \frac{Q_i\,(x-b_i)^n}{n!} + \sum \frac{q_i\,(x-c_i)^{n+1}}{(n+1)!} \tag{45.11}$$

which is the fundamental formula (I) in this book.

If $n > 0$, the application of formula (45.11) is simple. However, if $n < 0$, the expressions for higher moments cannot be composed by direct application of (45.11), because the number of terms decreases with the order of the moment.

Let us write the expressions for moments M_1, M_0 and M_{-1}

$$\left.\begin{aligned} M_1 = M_x &= \sum M_i + \sum Q_i\,(x-b_i) + \sum \frac{q_i\,(x-c_i)^2}{2!} \\ M_0 = Q_x &= \sum Q_i + \sum q_i\,(x-c_i) \\ M_{-1} = q_x &= \sum q_i \end{aligned}\right\} \tag{45.12}$$

Differentiation and integration of the higher moments results in

$$\frac{d}{dx}\left(\frac{M_n}{n!}\right)=\frac{M_{n-1}}{(n-1)!}\,;\quad \int\frac{M_n}{n!}\,dx=\frac{M_{n+1}}{(n+1)!} \tag{45.13}$$

Now the expressions for the integrals of the right-hand sides of equation (45.10) may be derived

$$\int M_1\,dx=\frac{M_{2X}}{2!} \tag{45.14a}$$

$$\int\left[\int M_1\,dx\right]dx=\frac{M_{3X}}{3!} \tag{45.14b}$$

$$\begin{aligned}\int M_1^2\,dx &= M_X\frac{M_{2X}}{2!}-Q_X\frac{M_{3X}}{3!}+q_X\frac{M_{4X}}{4!}-\\ &-\left[\sum M_i\frac{M_{2i}}{2!}-\sum Q_i\frac{M_{3i}}{3!}+\sum q_i\frac{M_{4i}}{4!}\right]\end{aligned} \tag{45.15a}$$

$$\begin{aligned}\int\left[\int M_1^2\,dx\right]dx &= M_X\frac{M_{3X}}{3!}-2Q_X\frac{M_{4X}}{4!}+3q_X\frac{M_{5X}}{5!}-\\ &-\left[\sum M_i\frac{M_{3i}}{3!}-2\sum Q_i\frac{M_{4i}}{4!}+3\sum q_i\frac{M_{5i}}{5!}\right]-\\ &-\left[\sum M_i\frac{M_{2i}}{2!}(x-x_i)-\sum Q_i\frac{M_{3i}}{3!}(x-x_i)+\right.\\ &\left.+\sum q_i\frac{M_{4i}}{4!}(x-x_i)\right]\end{aligned} \tag{45.15b}$$

$$\begin{aligned}\int M_1^3\,dx &= M_X^2\frac{M_{2X}}{2!}-2Q_XM_X\frac{M_{3X}}{3!}+\\ &+2(q_XM_X+Q_X^2)\frac{M_{4X}}{4!}-6q_XQ_X\frac{M_{5X}}{5!}+6q_X^2\frac{M_{6X}}{6!}-\\ &-\left[\sum(M_i'+M_i'')M_i\frac{M_{2i}}{2!}-2\sum(Q_i''M_i+M_i'Q_i)\frac{M_{3i}}{3!}+\right.\\ &+2\sum(q_i''M_i+M_i'q_i)\frac{M_{4i}}{4!}+2\sum(Q_i'+Q_i'')Q_i\frac{M_{4i}}{4!}-\\ &\left.-6\sum(q_i''Q_i+Q_i'q_i)\frac{M_{5i}}{5!}+6\sum(q_i'+q_i'')q_i\frac{M_{6i}}{6!}\right]\end{aligned} \tag{45.16a}$$

$$\begin{aligned}\int\left[\int M_1^2\,dx\right]dx &= M_X^2\frac{M_{3X}}{3!}-4Q_XM_X\frac{M_{4X}}{4!}+\\ &+6(q_XM_X+Q_X^2)\frac{M_{5X}}{5!}-\\ &-24q_XQ_X\frac{M_{6X}}{6!}+30q_X^2\frac{M_{7X}}{7!}-\left[\sum(M_i'+M_i'')M_i\frac{M_{3i}}{3!}-\right.\\ &-4\sum(Q_i''M_i+M_i'Q_i)\frac{M_{4i}}{4!}+6\sum(q_i''M_i+M_i'q_i)\frac{M_{6i}}{6!}+\end{aligned}$$

$$
\begin{aligned}
&+6\sum(Q'_i+Q''_i)Q_i\frac{M_{5i}}{5!}-24\sum(q''Q_i+Q'_iq_i)\frac{M_{6i}}{6!}+\\
&+30\sum(q'_i+q''_i)q_i\frac{M_{7i}}{7!}\Big]-\Big[\sum(M'_i+M''_i)M_i\frac{M_{2i}}{2!}(x-x_i)-\\
&-2\sum(Q''_iM_i+M'_iQ_i)\frac{M_{3i}}{3!}(x-x_i)+\\
&+2\sum(q''_iM_i+M'_iq_i)\frac{M_{4i}}{4!}(x-x_i)+\\
&+2\sum(Q'_i+Q''_i)Q_i\frac{M_{4i}}{4!}(x-x_i)-\\
&-6\sum(q''_iQ_i+Q'_iq_i)\frac{M_{5i}}{5!}(x-x_i)+\\
&+6\sum(q'_i+q''_i)q_i\frac{M_{6i}}{6!}(x-x_i)\Big]
\end{aligned}
\qquad (45.16b)
$$

In these formulas the quantities $\frac{M_{2X}}{2!}$, $\frac{M_{3X}}{3!}$, $\frac{M_{4X}}{4!}$, etc. are, correspondingly, the second, third, fourth, etc. moments in section X with abscissa x (Fig. 45.3), in which the linear and angular displacements are to be determined; M_X, Q_X, q_X are the bending moment, the transverse force and the intensity of the uniform continuous load in the same section; M_i, Q_i, q_i are the increments of these quantities at the initial point of section i (the beam is divided into sections according to the points of load application, and only those to the left of the cross section X are considered). $\frac{M_{2i}}{2!}$, $\frac{M_{3i}}{3!}$, $\frac{M_{4i}}{4!}$ etc. are moments of the second, third, fourth, etc. order at the initial point of section i. M'_i, Q'_i, q'_i are the bending moment, the transverse force and the intensity of the uniform continuous load to the left of the initial point of section i, that is, at the end of section $i-1$. M''_i, Q''_i, q''_i are the bending moment, the transverse force and the intensity of the uniform continuous load to the right of the initial

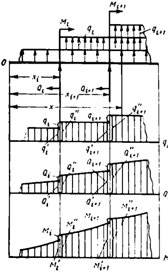

Fig. 45.3

point of section i; x_i is the distance from the origin of the coordinates to the initial point of section i.

Formulas (45.15) may be applied to the entire integrated section if the bending moment does not change its sign within this section; otherwise the integration must be carried out along separate sections within which the sign of the moment does not change.

In conclusion, we shall point out that although the formulas seem very complicated, since they take into account various combinations of loadings, in particular cases they are much simpler.

Problem 1. Determine the deflection and the rotation angle in the section A of a cantilever (Fig. 45.4) if the stress-strain relationship is

$$\sigma = K\varepsilon^{1/2}$$

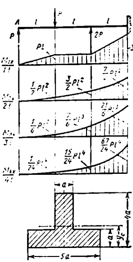

Fig. 45.4

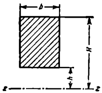

Fig. 45.5

The position of the neutral axis will be determined from formula (45.3). For a rectangle (Fig. 45.5).

$$I_{zm} = \frac{b}{m+1} \cdot (H^{m+1} - h^{m+1})$$

If the central axis is taken for the reference axis in the given section, then equation (45.3) is written as

$$\frac{a\,(2.75a+\gamma)^{3/2}}{3/2} + \frac{5a\,(1.25a-\gamma)^{3/2}}{3/2} - \frac{4a\,(0.25a-\gamma)^{3/2}}{3/2} = 0$$

hence, $\gamma = 0.2295a$.

The value of γ will be now determined by using the first three terms of the series (45.6).

$I^{h_1}_{z\,m-n}$ and $I^{h_2}_{z\,m-n}$ have the following values

$$I^{h_1}_{z\,0.5} = 3.0402^{2.5}; \qquad I^{h_2}_{z\,0.5} = 4.3251^{2.5}$$

$$I^{h_1}_{z-0.5} = 3.3166^{1.5}; \qquad I^{h_2}_{z-0.5} = 7.1803^{1.5}$$

$$I^{h_1}_{z-1.5} = -1.2061^{0.5}; \quad I^{h_2}_{z-1.5} = 7.0558^{0.5}$$

Substitution of these values into equation (45.6) results in

$$\gamma^2 + 5.0821a\gamma - 1.2442a^2 = 0$$

hence, $\gamma = 0.2341a$.

Thus, three terms of the series give an error in determining γ which amounts to about 2%.

After appropriate calculations we find that $D = 67.789K^2a^7$, when the displacement of the neutral axis is taken into account, and $D = 71.577K^2a^7$ otherwise.

This means that in this particular case the correction in rigidity γ for the displacement of the neutral axis alters the result by 6%; therefore, this effect need not be considered. However, this is not the case for all values of m; for example, for the same cross section when $m = 1/4$ this correction in D is equal to about 16.5%.

Let us determine the displacements at the end of the cantilever

$$D\frac{d^2V_X}{dx^2} = M_X^z$$

$$D\frac{dV_X}{dx} = D\Theta_X = D\Theta_0 + \int M_X^z\,dx$$

$$DV_X = DV_0 + D\Theta_0 x + \int\left[\int M_X^z\,dx\right]dx$$

where Θ_0, V_0 = rotation angle and deflection in the initial section of the beam, (Fig. 45.4)

Θ_X, V_X = rotation angle and deflection in the section at a distance x from the origin

When $x = 3l$, $\Theta_B = 0$ and $V_B = 0$,

$$D\Theta_0 = -\int\limits_{x=3l} M_X^z\,dx$$

$$DV_0 = -D\Theta_0\cdot 3l - \int\limits_{x=3l}\left[\int M_X^z\,dx\right]dx$$

$$\int\limits_{x=3l} M_X^z\,dx = -\Big[(-3Pl)\left(\frac{7}{2}Pl^2\right) - (-P)\left(\frac{1}{6}Pl^3\right) -$$

$$-(2P)\left(\frac{7}{6}Pl^3\right) - (-2P)\left(\frac{21}{6}Pl^3\right)\Big] = \frac{17}{3}P^2l^3$$

$$D\Theta_0 = -\frac{17}{3} P^2 l^3$$

$$\int\limits_{x=3l}\left[\int M_x^3\,dx\right]dx = -\Big[(-3Pl)\left(\frac{21}{6} Pl^3\right) -$$
$$-2(-P)\left(\frac{1}{24} Pl^4\right) - 2(2P)\left(\frac{15}{24} Pl^4\right) -$$
$$-2(-2P)\left(\frac{67}{24} Pl^4\right)\Big] - \Big[-(-P)\left(\frac{1}{6} Pl^3\right)2l -$$
$$-(2P)\left(\frac{7}{6} Pl^3\right) l\Big] = \frac{15}{4} P^2 l^4$$

$$DV_0 = \frac{53}{4} P^2 l^4$$

Problem 2. Prove that the beam (Fig. 45.6) is statically indeterminate and find its deflection in section B if the stress-strain relationship is

$$\sigma = K\varepsilon^{1/3}$$

The differential equation of the deflection curve of the beam is

$$D\frac{d^2V_x}{dx^2} = M_x$$

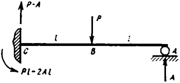

Fig. 45.6

For the problem under consideration $V_A = 0$ when $x = 2l$, therefore

$$\int\left[\int\limits_{x=2l} M_x^3\,dx\right]dx =$$
$$= -\Big[-4\sum(Q_i' M_i - M_i' Q_i)\frac{M_{4l}}{4!} + 6\sum(Q_i' + Q_i)\frac{M_{5l}}{5!}\Big] -$$
$$-\Big[-2\sum(Q_i' M_i + M_i' Q_i)\frac{M_{3l}}{3!}(2l - x_i) +$$
$$+2\sum(Q_i' + Q_i'')Q_i\frac{M_{4l}}{4!}(2l - x_i)\Big] = 0$$

Substitution of appropriate values in this equation results in

$$128\alpha^3 - 147\alpha^2 + 62\alpha - 9 = 0$$

hence

$$\alpha = \frac{A}{P} = 0.325 \quad \text{or} \quad A = 325P$$

The solution of this problem for a beam following Hooke's law (i.e., $m = 1$) gives $A = 0.313P$; thus, solutions for $m = 1$ and $m =$

$= 1/3$ do not differ significantly. The bending moment diagrams practically also coincide.

The deflection at point B is determined from

$$DV_B = M_B^2 \frac{M_{3B}}{3!} - 4Q_B M_B \frac{M_{4B}}{4!} + 6Q_B^2 \frac{M_{5B}}{5!} = -0.00458P^3l^5$$

For many statically indeterminate beams the non-linearity of the stress-strain relationship does not lead to any substantial redistribution of support reactions. That is why if particularly high precision is not required the unknown values in statically indeterminate systems may be found by assuming $m = 1$, while deflections should be calculated by formulas (45.14)-(45.16).

If the deflection is to be found at a certain point of the span, then Mohr's integral may be used

$$D\Delta = \int M^{1/m} M_1 \, dx$$

where M_1 = moment induced by a unit factor applied at the point under consideration

Δ = unknown deflection

Displacements may be determined by means of Castigliano's generalized theorem, but if the problem is to establish the equation of the flexural curve or to determine deflections at several points, then the described method based on the application of higher moments may be recommended.

Chapter IX

Application of Moment-Operational Method to Structural Mechanics of Ships

46. FLEXURE OF IRREGULAR DECKS

There are several particularly difficult problems in designing decks which involve such phenomena as deformation due to shear, pliancy of the bearing contour, variable bending and shear stiffness, elasto-plastic behaviour. Meanwhile the precise calculation of such structures, which are widely used in engineering, is of vital necessity, because it makes the strength design more reliable and allows to use still concealed bearing capacity.

Deck design problems dealing with structural and loading irregularities can be effectively solved by the moment-operational method, as shown by solving of many engineering problems in the previous sections.

Let us examine an irregular deck with an arbitrary number of transverse beams and longitudinal braces with arbitrary boundary conditions (Fig. 46.1).

l_{ji}, I_{ji}, ω^*_{ji}, P_{ji} = length, moment of inertia, cross-sectional area and load on brace j between transverse beams $(i - 1)$ and i

B_i, $I_i(y)$, $\omega^*_i(y)$, Q_i = same for transverse beam

z_{ji} = ordinate of arbitrary section of brace j between tranverse beams $(i - 1)$ and i

R_{ji} = interaction at the point of intersection of brace j and beam i

It will be shown further that the method outlined here may be applied to arbitrarily distributed rigidity.

However, to simplify the calculations we assume that the rigidity of the bracing is constant within each separate span between the transverse beams.

According to the moment-operational method, deflections between beams $(i-1)$ and i are given by

$$\left.\begin{aligned} &EI_{ji}V_j^{IV}(z_{ji})=M_{-1j}(z_{ji});\quad EI_{ji}V_j^{III}(z_{ji})=M_{0j}(z_{ji})\\ &EI_{ji}V_j^{II}(z_{ji})=M_{1j}(z_{ji});\quad EI_{ji}V_j^{I}(z_{ji})=\frac{M_{2j}(z_{ji})}{2!}\\ &EI_{ji}V_j(z_{ji})=\frac{M_{3j}(z_{ji})}{3!} \end{aligned}\right\} \tag{46.1}$$

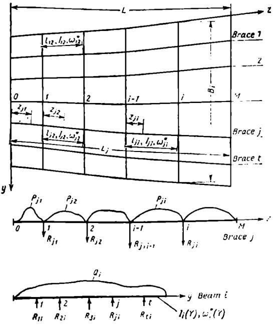

Fig. 46.1

Moments of the n-th order are determined by the fundamental formula (III) of this book

$$\frac{M_{nj}(z_{ji})}{n!}=EI_{ji}V_j(l_{j,i-1})\frac{z_{ji}^{n-3}}{(n-3)!}+EI_{ji}\Theta_j(l_{j,i-1})\frac{z_{ji}^{n-2}}{(n-2)!}+$$

$$+M_j(l_{j,i-1})\frac{z_{ji}^{n-1}}{(n-1)!}+M_j(l_{j,i-1})\frac{z_{ji}^{n}}{n!}+\frac{M_{nj}(P_{ji},z_{ji})}{n!} \tag{46.2}$$

where $\frac{M_{nj}(P_{ji}, z_{ji})}{n!}$ represents the moment of the *n*-th order of the load P_{ji} in the span between beams $(i-1)$ and i. By using the fundamental formula (I) and formulas (46.1) and (46.2), the following expression is derived for the moment of the *n*-th order

$$\frac{M_{nj}(z_{ji})}{n!} = \sum_{r=0}^{3} \frac{M_{rj}(l_{j,\,i-1})}{r!} \cdot \frac{\beta_{ji}^{r}}{r!} \cdot \frac{z_{ji}^{n-r}}{(n-r)!} + \frac{M_{nj}(P_{ji}, z_{ji})}{n!} \qquad (46.3)$$

The conversion coefficient $\frac{\beta_{ji}^{r}}{r!}$ should be assumed equal to

$$\frac{\beta_{ji}^{r}}{r!} = \begin{cases} 1, & \text{when } r = 0,\ 1 \\ \dfrac{I_{ji}}{I_{j,\,i-1}}, & \text{when } r = 2,\ 3 \end{cases} \qquad (46.4)$$

The shear stiffness of the bracing is assumed infinite in the derivation of formulas (46.1)-(46.3).

If the shear stiffness of the bracing is finite, the last term of formula (46.1) becomes

$$EI_{ji}V_{ji} = \frac{M_{3j}(z_{ji})}{3!} - \frac{EI_{ji}}{G\omega_{ji}^{*}}\left[M_{1j}(z_{ji}) - M_{1j}(l_{j,\,i-1})\right] \qquad (46.4a)$$

while the other terms remain the same.

Assuming

$$\gamma_{ji} = \frac{EI_{ji}}{G\omega_{ji}^{*}} \qquad (46.5)$$

from formulas (46.3) and (46.4*a*) we obtain

$$\frac{M_{nj}(z_{ji})}{n!} = \sum_{r=0}^{3} \frac{M_{rj}(s_{j,\,i-1})}{r!} \cdot \frac{\beta_{ji}^{r}}{r!} \cdot \frac{z_{ji}^{n-r}}{(n-r)!} \cdot \frac{\delta_{0ji}^{n-r}}{(n-r)!} +$$

$$+ \frac{M_{nj}(P_{ji}z_{ji})}{n!} \cdot \frac{\delta_{1ji}^{n}}{n!} \qquad (46.6)$$

Symbols $\frac{\delta_{0ji}^{k}}{k!}$ and $\frac{\delta_{1ji}^{k}}{k!}$ in this formula designate the following conditions governing the calculation of terms $\frac{z_{ji}^{k}}{k!}$ and $\frac{M_{kj}(P_{ji}z_{ji})}{k!}$;

$$\frac{z_{ji}}{3!} - \gamma_{ji}z_{ji} = \frac{z_{ji}^{3}}{3!} \cdot \frac{\delta_{0ji}^{3}}{3!} \qquad (46.7)$$

$$\frac{M_{3j}(P_{ji}z_{ji})}{3!} - \gamma_{ji}\frac{M_{1j}(P_{ji}z_{ji})}{1!} = \frac{M_{3j}(P_{ji}z_{ji})}{3!} \cdot \frac{\delta_{1ji}}{3!} \qquad (46.8)$$

and $\frac{\delta_0^k}{k!} = \frac{\delta_1^k}{k!} = 1$ when $k \neq 3$, i.e.,

$$\left.\begin{aligned} \frac{\delta_{0ji}^k}{k!} &= \begin{cases} 1, \text{ when } k \neq 3 \\ 0 \text{ when } k = 3 \end{cases} \text{ according to (46.7)} \\ \frac{\delta_{1ji}^3}{k!} &= \begin{cases} 1, \text{ when } k \neq 3 \\ 0 \text{ when } k = 3 \end{cases} \text{ according to (46.8)} \end{aligned}\right\} \tag{46.9}$$

Expression (46.9) makes it possible to design a structure with shear deformations by using the same relationships (46.3) derived without the shear taken into account.

To evaluate the redundancy of the deck we assume that the right end of the span $(i-1)$, i has a coordinate $z_{ji} = l_{ji} - 0$ (Fig. 46.1); then the reaction R_{ji} will not be included in the loading of this span. Therefore, according to formulas (46.2) and (46.6), the moment of the n-th order for the j-th brace is expressed by

$$\frac{\overline{M}_{nj}(z_{ji})}{n!} = \frac{M_{nj}(z_{ji})}{n!} + R_{j,\, i-1} \frac{z_{ji}^n}{n!} \cdot \frac{\delta_{0ji}^n}{n!} \tag{46.10}$$

The unknown reaction R_{ji} may be eliminated by means of relations derived from examining the behaviour of the transverse beams. The i-th beam is shown in Fig. 46.1*c*. The well-known solutions for such a beam may be used to compose the expression for the reaction in the point of intersection of beam i and brace j

$$R_{j,\, i} = S_{j,\, 0,\, i} Q_i + \sum_{q=1}^{t} S_{j,\, q,\, i} E I_{qi} V_q (l_{qi}) \tag{46.11}$$

where $S_{j,\,0,\,i}$ = influence factor of load Q_i upon reaction at point j for beam i

$S_{j,\,q,\,i}$ = influence factor of deflection of beam i at point q upon reaction at point j.

After substitution from formulas (46.11) and (46.10) with regard to (46.1) we obtain

$$\begin{aligned} \frac{\overline{M}_{nj}(z_{ji})}{n!} = \sum_{r=0}^{3} & \frac{\overline{M}_{rj}(l_{j,\, i-1})}{r!} \cdot \frac{\beta_{ji}^r}{r!} \cdot \frac{z_{ji}^{n-r}}{(n-r)!} \cdot \frac{\delta_{0ji}^{n-r}}{(n-r)!} + \\ & + \frac{M_{nj}(P_{ji} z_{ji})}{n!} \cdot \frac{\delta_{1ji}^n}{n!} + \frac{z_{ji}^n}{n!} \cdot \frac{\delta_{0ji}^n}{n!} S_{j,\, 0,\, i-1} Q_{i-1} + \\ & + \frac{z_{ji}^n}{n!} \cdot \frac{\delta_{0ji}^n}{n!} \sum_{q=1}^{t} S_{j,\, q,\, i-1} \frac{\overline{M}_{3q}(l_{q,\, i-1})}{3!} \ldots \end{aligned} \tag{46.12}$$

This formula expresses moments of order n in section i in terms of the n-th moments in any arbitrary section. Let us express these moments in section i through moments $\frac{M_{rj}(0)}{r!}$ in section $Z = 0$.

First, the expression for the n-th moment will be composed for various values of

$$z = z_1,\ z_2,\ z_3\ \ldots$$

For example, if $z_l = z_1$, according to (46.6)

$$\frac{M_{nj}(z_{j1})}{n!} = \sum_{r=0}^{3} \frac{M_{rj(0)}}{r!} \cdot \frac{\beta_{j1}^{r}}{r!} \cdot \frac{z_{j1}^{n-r}}{(n-r)!} \cdot \frac{\delta_{0j1}^{n-2}}{(n-r)!} + \frac{M_{nj}(P_{j1}, z_{j1})}{n!} \cdot \frac{\delta_{1j1}^{n}}{n!}$$

and

$$\frac{M_{3j}(z_{j1})}{3!} = \sum_{r=0}^{3} \frac{M_{rj(0)}}{r!} \cdot \frac{\beta_{j1}^{r}}{r!} \cdot \frac{z_{j1}^{3-r}}{(3-r)!} + \frac{M_{3j}(P_{j1}, z_{j1})}{3!} \cdot \frac{\delta_{1j1}^{3}}{3!} \qquad (43.13)$$

Let us determine the n-th moments for the second span, when $Z_1 = Z_2$. According to (46.12). and considering (46.13),

$$\begin{aligned}\frac{\overline{M}_{nj}(z_{jr})}{n!} = {} & \sum_{q=1}^{t}{}' \sum_{s=0}^{3}{}' \frac{M_{sq(0)}}{s!} \cdot \frac{A_{sq}^{n}(z_{jr})}{n!} + \\ & + \sum_{s=0}^{3} \frac{M_{sj}(P_{j1}, l_{j1})}{s!} \cdot \frac{\delta_{1j1}^{s}}{s!} \cdot \frac{b_{s}^{n}(z_{jr})}{n!} + \\ & + \sum_{q=1}^{t}{}' \frac{M_{3q}(P_{q1}, l_{q1})}{3!} \cdot \frac{\delta_{1q1}^{3}}{3!} \cdot \frac{C_{q1}^{n}(z_{jr})}{n!} + \\ & + Q\frac{D_{1}^{n}(z_{jr})}{n!} + \frac{M_{nj}(P_{jr}, z_{0r})}{n!} \cdot \frac{\delta_{1jr}^{n}}{n!}\end{aligned} \qquad (46.14)$$

where

$$\left.\begin{aligned}\frac{M_{nj}(z_{j1})}{n!} &= \frac{\overline{M}_{nj}(z_{j1})}{n!} \\ \frac{A_{sq}^{n}(z_{jr})}{n!} &= S_{jql}\frac{\beta_{j1}^{s}}{s!} \cdot \frac{l_{q1}^{3-s}}{(3-s)!} \cdot \frac{\delta_{0q1}^{3-s}}{(3-s)!} \cdot \frac{z_{jr}^{n}}{n!} \cdot \frac{\delta_{0jr}^{n}}{n!} + \\ &+ e_{jq}\sum_{r=0}^{3} \frac{\beta_{jr}^{r}}{r!} \cdot \frac{z_{jr}^{n-r}}{(n-r)!} \cdot \frac{\delta_{0jr}^{n-r}}{(n-r)!} \cdot \frac{l_{j1}^{r-s}}{(r-s)!} \cdot \frac{\delta_{0j1}^{r-s}}{(r-s)!} \\ \frac{b_{s}^{n}(z_{jr})}{n!} &= \frac{\beta_{jr}^{s}}{s!} \cdot \frac{z_{rr}^{n-s}}{(n-s)!}\,\frac{\delta_{0jr}^{n-s}}{(n-s)!} \\ \frac{C_{q1}^{n}(z_{jr})}{n!} &= S_{jq1}\frac{z_{rr}^{n}}{n!} \cdot \frac{\delta_{0rr}^{n}}{n!} \\ \frac{D_{1}^{n}(z_{jr})}{n!} &= S_{j01}\frac{z_{rr}^{n}}{n!} \cdot \frac{\delta_{0jr}^{n}}{n!}\end{aligned}\right\} \qquad (46.15)$$

The symbol e_{jq} in these formulas has the following meanings

$$e_{jq} = \begin{cases} 1, & \text{when } j = q \\ 0, & \text{when } j \neq q \end{cases} \qquad (46.16)$$

Now expression will be composed for the n-th moment in section Z_i when $i \geqslant 4$ (these expressions will be used further)

$$\begin{aligned}
\frac{\overline{M}_{nj}(z_{ji})}{n!} = {} & \sum_{q=1}^{i} \sum_{s=0}^{3} \frac{M_{sq}(0)}{s!} \cdot \frac{A^n_{sq}(z_{ji})}{n!} + \\
& \sum_{k=1}^{i-r} \sum_{q=1}^{i} \sum_{s=0}^{3} \frac{M_{sq}(P_{qk}, l_{qk})}{s!} \cdot \frac{\delta^s_{1qk}}{s!} \cdot \frac{m^n_{sqk}(z_{ji})}{n!} + \\
& + \sum_{k=1}^{i-1} Q_k \frac{D^n_k(z_{ji})}{n!} + \sum_{s=0}^{3} \frac{M_{sj}(P_{j,\,i-1}, l_{j,\,i-1})}{s!} \times \\
& \times \frac{\delta^s_{1jj,\,i-1}}{s!} \cdot \frac{b^n_s(z_{ji})}{n!} + \sum_{q=1}^{i} \frac{M_{3q}(P_{q,\,i-1}, l_{q,\,i-1})}{3!} \times \\
& \times \frac{\delta^3_{1,\,q,\,i-1}}{3!} \cdot \frac{C^n_{1,\,q,\,i-1}(z_{ji})}{n!} + \frac{M_{nj}(P_{ji}, z_{ji})}{n!} \cdot \frac{\delta^n_{1ji}}{n!}
\end{aligned} \qquad (46.17)$$

where

$$\begin{aligned}
\frac{A^n_{sq}(z_{ji})}{n!} = {} & \sum_{r=0}^{3} \frac{A^r_{sq}(l_{j,\,i-1})}{r!} \cdot \frac{\beta^r_{ji}}{r!} \cdot \frac{z^{n-r}_{ji}}{(n-r)!} \cdot \frac{\delta^{n-r}_{0ji}}{(n-r)!} + \\
& + \sum_{k=1}^{i} S_{j,\,k,\,i-1} \frac{A^3_{sq}(l_{k,\,i-1})}{3!} \cdot \frac{z^n_{ji}}{n!} \cdot \frac{\delta^n_{0ji}}{n!}
\end{aligned}$$

$$\left.\frac{m^n_{sqk}(z_{ji})}{n!}\right|_{i \geqslant k+3} =$$

$$\begin{aligned}
= {} & \sum_{r=0}^{3} \frac{m^n_{sqk}(l_{j,\,i-1})}{n!} \cdot \frac{\beta_{ji}}{r!} \cdot \frac{z^{n-r}_{ji}}{(n-r)!} \cdot \frac{\delta^{n-r}_{0ji}}{(n-r)!} + \\
& + \sum_{p=1}^{i} S_{j,\,p,\,i-1} \frac{m^3_{sqk}(l_{p,\,i-1})}{3!} \cdot \frac{z^n_{ji}}{n!} \cdot \frac{\delta^n_{0ji}}{n!}
\end{aligned}$$

$$\left.\frac{m^n_{sqk}(z_{ji})}{n!}\right|_{i=k+2} = \frac{B^n_{sqk}(z_{ji})}{n!} + e_{3s} \frac{C^{n\,\prime}_{qk}(z_{ij})}{n!}$$

$$\frac{B^n_{sqk}(z_{ji})}{n!} = S_{j,\,q,\,k+1} \frac{b^3_s(l_{q,\,i-1})}{3!} \cdot \frac{z^n_{ji}}{n!} \cdot \frac{\delta^n_{0ji}}{n!} +$$

$$\left.\begin{aligned}
&+\varepsilon_{jq}\sum_{r=0}^{3}\frac{\beta_{ji}^{r}}{r!}\cdot\frac{b_{s}^{r}(l_{j,i-1})}{r!}\cdot\frac{z_{ji}^{n-r}}{(n-r)!}\cdot\frac{\delta_{0ji}^{n-r}}{(n-r)!}\\
&\frac{C_{qk}^{n}(z_{ji})}{n!}=\sum_{r=0}^{3}\frac{C_{qk}^{r}(l_{j,i-1})}{r!}\cdot\frac{\beta_{ji}^{r}}{r!}\cdot\frac{z_{ji}^{n-r}}{(n-r)!}\cdot\frac{\delta_{0ji}^{n-r}}{(n-r)!}+\\
&\quad+\sum_{p=1}^{i}S_{j,p,i-1}\frac{C_{qk}^{3}(l_{p,i-1})}{3!}\cdot\frac{z_{ji}^{n}}{n!}\cdot\frac{\delta_{0ji}^{n}}{n!}\\
&\frac{b_{s}^{n}(z_{ji})}{n!}=\frac{\beta_{ji}^{s}}{s!}\cdot\frac{z_{ji}^{n-s}}{(n-s)!}\cdot\frac{\delta_{0ji}^{n-s}}{(n-s)!}\\
&\frac{C_{qk}^{n\,\prime}(z_{ji})}{n!}=S_{jqk}\frac{z_{ji}^{n}}{n!}\cdot\frac{\delta_{0ji}^{n}}{n!}\\
&\frac{D_{n}^{k}(z_{ji})}{n!}=\Big|_{i\geqslant k+r}=\\
&=\sum_{r=0}^{3}\frac{D_{k}^{r}(l_{j,i-1})}{r!}\cdot\frac{\beta_{ji}^{r}}{r!}\cdot\frac{z_{ji}^{n-r}}{(n-r)!}\cdot\frac{\delta_{0ji}^{n-r}}{(n-r)!}+\\
&\quad+\sum_{q=1}^{i}S_{j,q,i-1}\frac{D_{k}^{3}(l_{q,i-1})}{3!}\cdot\frac{z_{ji}^{n}}{n!}\cdot\frac{\delta_{0ji}^{n}}{n!}\\
&\frac{D_{k}^{n}(z_{ji})}{n!}\Big|_{i=k+1}=S_{j,0,i-1}\frac{z_{ji}^{n}}{n!}\cdot\frac{\delta_{0ji}^{n}}{n!}
\end{aligned}\right\}\quad(46.18)$$

Formulas (46.17) will now be used to compose expressions for the n-th moment $\frac{\overline{M}_{nj}(z_{j\mu})}{n!}$ in the support section $z_i=z_\mu$.

The boundary conditions will be satisfied for every brace j by moments of the n-th order when $z=0$. For the general case of arbitrary boundary conditions the following expressions are derived for brace j.

When $z_i=0$

$$\frac{1}{EI_{j1}}\cdot\frac{M_{2j(0)}}{2!}=\mathfrak{A}_{j0}\frac{M_{1j(0)}}{1!}+\frac{M_{1j(0)}\overline{M}_{1j}(l_{j\mu})}{L_jG\omega_{j1}^{*}}$$

and

$$\frac{1}{EI_{j0}}\cdot\frac{M_{3j(0)}}{3!}=-A_{j0}\frac{M_{0j(0)}}{0!}\qquad(46.19)$$

When $z_l = l\mu$

$$\frac{1}{EI_{j\mu}} \cdot \frac{M_{2j}(l_\mu)}{2!} = -\mathfrak{A}_{j\mu} \frac{M_{1j}(l_\mu)}{1!} + \frac{M_{1j(0)} - \overline{M}_{1j}(l_{j\mu})}{L_j G, \omega^*_{j\mu}}$$

$$\frac{1}{EI_{j\mu}} \cdot \frac{M_{3j}(l_\mu)}{3!} = A_{j\mu} \frac{M_{0j}(l_\mu)}{0!}$$

where $\mathfrak{A}_{j0}$, $\mathfrak{A}_{j\mu}$, A_{j0}, $A_{j\mu}$ are coefficients of elastic fixing and elastic pliancy of supports when $z = 0$ and $z = 00$ for brace j. These coefficients are known to be numerically equal to the rotation angles caused by a unit moment and to the deflection caused by a unit force.

The number of relationships of the form (46.19) depends on the number of braces; hence, unknown moments $\frac{M_{nj(0)}}{n!}$ are determined from a set of $4t$ equations.

Thus, the evaluation of the redundancy of decks is reduced to calculation of moments of the n-th order. This is performed by means of formulas (46.13), (46.14), (46.16), and (46.17) which are easily treated by digital computers.

This method may be applied as well to longitudinal beams of variable bending and shear stiffness; in this case the beam is divided into sections of constant rigidity, and the described method is applied to each section.

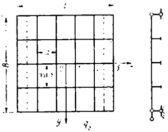

Fig. 46.2

For example, let us design the bottom plates of a cargo ship shell subjected to upward hydrostatic pressure. For simplicity, the intensity of the load q_0 is assumed constant over the entire surface of the bottom plates. The bottom is flat and has a constant width $B_i = B$ and length L; it consists of four transverse beams of equal spacing a and three longitudinal braces symmetrical about the longitudinal axis of the ship; the braces are equally spaced from each other and from the sides (Fig. 46.2). The transverse beams are simply supported at the sides, the longitudinal braces are fixed at the transverse bulkheads.

Three sets of assumptions will be considered to give three variants of design.

(a) Shear strain is disregarded; longitudinal braces are supported by transverse bulkheads, which are considered absolutely rigid supports.

(b) Shear strain in webs is taken into account; longitudinal braces are supported by transverse bulkheads, which are considered absolutely rigid supports.

(c) Shear strain in webs is taken into account; longitudinal braces are supported by transverse bulkheads, which are considered yielding supports (thus, the pliability of the bulkheads is taken into account).

Case (a) corresponds to conditions assumed in contemporary design of local strength of decks. The designing procedure is described in various manuals.

We shall carry out calculations with parameters $L = B$; $B = 5a$. The moment of inertia and the cross-sectional area of the central longitudinal brace (the keel) are given by I_{1i} and ω_{1i}^*, respectively. The same parameters for the longitudinal brace of the ship between the keel and the side (the stringer) are designated by I_{2i} and ω_{2i}^*.

Similarly, for the transverse beam we have $I_i(y) = I_0$ and $\omega_i^*(y) = \omega_0^*$ while the following relations exist

$$I_{1i} = I_1; \quad \omega_{1i}^* = \omega_1^*; \quad I_{2i} = I_2; \quad \omega_{2i}^* = \omega_2^*$$

$$\frac{I_1}{I_2} = 1; \quad \frac{\omega_1^*}{\omega_2^*} = 1.25; \quad \frac{I_1}{B^2\omega_1^*} = 0.015; \quad \frac{I_1}{I_0} = 2;$$

$$\frac{\omega_1^*}{\omega_0^*} = 2$$

The moment of inertia of the transverse bulkhead I_n and the cross-sectional area of its sheets ω_n^* are related by

$$\frac{I_n}{B^2\omega_n^*} = 0.015$$

The flexibility coefficients of the transverse bulkhead A_1 for the keel and A_2 for the stringer are determined by the following formula, which takes into account the bending, shear, and local compression of bulkheads at the points of intersection with the keel and the stringers

$$A_j = \frac{B^3}{48EI_n}\left(1 + \frac{125}{4}\cdot\frac{I_n}{B^2\omega_n^*}\right) + \frac{H}{EF_{pj}} \tag{46.20}$$

where H = depth of bulkhead, $H = 0.8$

F_{pj} = cross-sectional area of bulkhead compression strut in the plane of keel ($j = 1$) and in the plane of stringer ($j = 2$)

Substitution of the numerical values into formula (46.20) results in

$$A_1 = \frac{0.2a^3}{EI_1}; \quad A_2 = \frac{0.3a^3}{EI_1}$$

Therefore, the boundary conditions (46.19) may be derived from the symmetry of the construction and of the load as

$$\left.\begin{aligned} M_{2j}(0) &= 0 \\ M_{0j}(0) &= 0 \\ M_{2j}(l_{j3}) &= 0 \\ \frac{M_{3j}(l_{j3})}{3!} &= A_j EI_j \cdot \frac{M_{0j}(l_{j3})}{0!} \quad (j = 1,\ 2) \end{aligned}\right\} \tag{46.21}$$

The load is assumed to be transferred by the transverse beams to the longitudinal braces in the form of reactions. Hence, for all the spans of brace *j* except the last span,

$$\frac{M_{nj}(P_{ji},\ z_{ji})}{n!} = 0$$

Thus, in this particular case two longitudinal braces ($j = 1, 2$) are subjected to loads Q_i transferred by the transverse beams

$$Q_i = aq_0B = Q$$

Due to the symmetry of the construction and loads with respect to the middle transverse axis of the bottom plates, there will be three spans to design, the first of length $l_{01} = 0.5a$, the second and the third of length $l_{12} = l_{23} = a$.

For the end span *2-3* part of the load is transmitted from the strip 0.5*a* wide (the shaded area in Fig. 46.2) to the longitudinal braces. According to formula (46.12), the bending moment due to the external loads for this span is given by

$$\frac{M_{n1}(Z_{13})}{n!} = q_0 \frac{B}{4} \cdot \frac{\left(z_{13} - \frac{a}{2}\right)^{n+1}}{(n+1)!} = \frac{M_{n2}(Z_{23})}{n!}$$

The final results obtained for the three sets of assumptions (*a*, *b* and *c*) are presented in Table 46.1.

Table 46.1

Brace	Keel				Stringer			
Flexure elements / Variants	$\dfrac{V_{\max}}{\frac{Qa^3}{EI_1}}$	$\dfrac{M_{span}}{Qa}$	$\dfrac{M_{sup}}{Qa}$	$\dfrac{N_{\max}}{Q}$	$\dfrac{V_{\max}}{\frac{Qa^3}{FI_1}}$	$\dfrac{M_{span}}{Qa}$	$\dfrac{M_{sup}}{Qa}$	$\dfrac{N_{\max}}{Q}$
a	0.464	−0.288	0.565	0.660	0.342	−0.209	0.445	0.573
b	1.147	−0.244	0.535	0.631	0.926	−0.167	0.371	0.467
c	1.267	−0.247	0.530	0.633	1.014	−0.157	0.340	0.440

It follows from the table that the shear strain exerts substantial influence on the parameters of flexure. The flexibility of the transverse bulkheads does not influence these parameters as much.

In conclusion we can say that the method of calculation discussed here enables calculation of a compartment as a whole without splitting it into separate parts, i.e., permits a three-dimensional calculation of a compartment. In particular, it is quite easy to evaluate the combined bending of the deck and the bottom plates bound together by the bracket beams. In this case the load P_{ji} must represent the interaction of two deckings.

Deckings with half-bulkheads, which are usually designed approximately, may be calculated more precisely by the moment-operational method.

47. NON-UNIFORM BEAMS ON ELASTIC FOUNDATION

Integration of the differential equation of flexure for beams on non-uniform elastic foundation is connected with intricate calculations. The moment-operational method simplifies the problem considerably. Although the solution in this case will be only approximate, the results are sufficiently accurate.

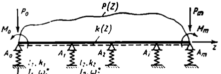

Fig. 47.1

Let us examine the bending of a non-prismatic beam on elastic foundation of variable rigidity and on elastic supports, the elastic properties of which are given by flexibility coefficients A_1, A_2, ..., A_m (Fig. 47.1).

For a uniform beam with moment of inertia I resting on an elastic foundation with rigidity coefficient $k = EI\beta^2$ the moment-operational method gives the following expressions for calculating the parameters of flexure

$$\left.\begin{aligned} EIV &= \frac{M_3}{3!} - \beta^2 \frac{M_7}{7!} + \beta^4 \frac{M_{11}}{11!} - \beta^6 \frac{M_{15}}{15!} + \dots \\ EIV' &= \frac{M_2}{2!} - \beta^2 \frac{M_6}{6!} + \beta^4 \frac{M_{10}}{10} - \beta^6 \frac{M_{14}}{14!} + \dots \\ EIV'' &= \frac{M_1}{1} - \beta^2 \frac{M_5}{5!} + \beta^4 \frac{M_9}{9!} - \beta^6 \frac{M_{13}}{13} + \dots \\ EIV''' &= \frac{M_0}{0'} - \beta^2 \frac{M_4}{4!} + \beta^4 \frac{M_8}{8!} - \beta^6 \frac{M_{12}}{12!} + \dots \end{aligned}\right\} \qquad (47.1)$$

where the moment of the n-th order is expressed by formula (46.2).
From (47.1) and (46.2) by using formula (I) we derive

$$\frac{M_n(z)}{n!} = \sum_{r=0}^{3} \frac{M_{r(0)}}{r!} \varphi_{n-r}(z) + \frac{M_n(p)}{n!} F_n(z) \qquad (47.2)$$

Functions $\varphi_{n-r}(z)$ and $F_n(z)$ for a uniformly distributed load were given by N. Soldatov, who ascertained that these functions might be determined by means of Puzyrevsky's functions generally used in solving prismatic beams on elastic foundation.

The expression for the moment $\frac{M_n(p)}{n!} F_n(z)$ may be derived from (47.1) and (46.2); in particular, for a uniformly distributed load P_0

$$\frac{M_n(p)}{n!} F_n(z) = P_0 \varphi_{n+1}(z) \qquad (47.3)$$

Functions $\varphi_i(z)$ are interconnected by the same relations as Puzyrevsky's functions, i.e., during differentiation with respect to z one function transforms into another according to the law

$$\frac{d}{dz} \varphi_n(z) = \varphi_{n-1}(z) \qquad (47.4)$$

The shear strain may be taken into account in the same way as when deriving (46.6). After introducing conventional symbols $\frac{\delta_{ri}^n}{n!}$ which represent certain rules for determining the quantity to the left of this symbol, formula (47.2), allowing for shear will have the form

$$\frac{M_n(z)}{n!} = \sum_{r=0}^{3} \frac{M_{r(0)}}{r!} \varphi_{n-r}(z) \frac{\delta_{r0}^n}{n!} + \frac{M_n(p)}{n!} F_n(z) \frac{\delta_i^n}{n!} \qquad (47.5)$$

where the symbols are determined from the following expressions

$$\varphi_{n-r}(z) \frac{\delta_{r0}^n}{n!} = \varphi_{n-r}(z) - \gamma \varphi_{n-r-2}(z), \quad \text{when } n = 3,\ r \neq 1$$

$$\varphi_{n-r}(z) \frac{\delta_{r0}^n}{n!} = \varphi_{n-r}(z), \ \text{otherwise} \qquad (47.6)$$

The meaning of the symbol $\frac{\delta_i^n}{n!}$ may be shown, as an example, for the moment of a uniformly distributed load expressed by (47.3)

$$\frac{M_n(P)}{n!} F_n(z) \frac{\delta_i^n}{n!} = P_0 [\varphi_{n+1}(z) - \gamma \varphi_{n-1}(z)], \ \text{when } n = 3$$

$$\frac{M_n(P)}{n!} F_n(z) \frac{\delta_i^n}{n!} = P_0 \varphi_{n+1}(z), \ \text{otherwise} \qquad (47.7)$$

Formula (47.5) will serve as a basis for solving the problem discussed in this section.

We divide the given beam into a number of spans within which the moment of inertia I_i, the cross-sectional area ω_i^*, the rigidity of elastic foundation k_i and the load P_i can be assumed constant.

Then, from formula (47.5) we write

$$\frac{M_n(z_i)}{n!}=\sum_{r=0}^{3}\frac{\beta_i^r}{r!}\,\frac{M_r(l_{i-1})}{r!}\cdot\varphi_{n-r}(z_i)\,\frac{\delta_{r0}^n}{n!}+P_i\varphi_{n+1}(z_i)\,\frac{\delta_i^n}{n!} \qquad (47.8)$$

As in the previous section we assume that the right-hand end of span $(i-1)$, i coincides with section $z_i=b_i-0$; therefore, relationship (47.8) holds for those spans $(i-1)$, i, in which the section $(i-1)$ does not contain reactions R_{i-1}. For spans with the unknown reaction R_{i-1} at the beginning relationship (47.8) acquires the form

$$\frac{\overline{M}_n(z_i)}{n!}=\frac{M_n(z_i)}{n!}+R_{i-1}\varphi_n(z_i)\,\frac{\delta_{n0}^n}{n!} \qquad (47.9)$$

The unknown reaction R_{i-1} will be determined from the condition that the deflections of the beam and support at the point of their connection are equal. The deflections will be considered positive if the points move downwards, i.e., in the positive direction of the Y axis. Positive directions of shear forces and bending moments are shown in Fig. 47.2.

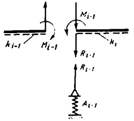

Fig. 47.2

From the previous discussion the following relationship between the reaction and the deflection of support $(i-1)$ may be derived

$$R_{i-1}=-\frac{V_{i-1}}{A_{i-1}}$$

hence

$$R_{i-1}=-\frac{1}{EI_iA_{i-1}}\cdot\frac{M_3(z_i)}{3!}\bigg|_{z_i=0} \qquad (47.10)$$

or, with regard to formula (47.8)

$$R_{i-1}=-\frac{1}{EI_iA_{i-1}}\left[\sum_{r=0}^{3}\frac{\beta_i^r}{r!}\cdot\frac{M_r(l_{i-1})}{r!}\varphi_{3-r}(z_i=0)\,\frac{\delta_{r0}^3}{3!}+\right.$$

$$\left.+P_i\varphi_4(z_i=0)\,\frac{\delta_i^3}{3!}\right]$$

Substituting this expression into formula (47.9) and denoting

$$\left.\begin{aligned}\varphi_{n-r}(z_l)\frac{\delta_{r0}^{n}}{n!}-\frac{1}{EI_lA_{l-1}}\varphi_{3-r}(z_l=0)\frac{\delta_{r0}^{3}}{3!}\varphi_n(z_l)\frac{\delta_{00}^{n}}{n!}=\overline{\varphi}_{n-r}(z_l)\\ \varphi_{n+1}(z_i)\frac{\delta_{1}^{n}}{n!}-\frac{1}{EI_lA_{l-1}}\varphi_4(z_i=0)\frac{\delta_{1}^{3}}{3!}\varphi_n(z_l)\frac{\delta_{00}^{n}}{n!}=\overline{\varphi}_{n+1}(z_l)\end{aligned}\right\}\qquad(47.11)$$

we obtain

$$\frac{\overline{M}_n(z_l)}{n!}=\sum_{r=0}^{3}\frac{\beta_i^r}{r!}\,\frac{M_r(l_l-1)}{r!}\,\overline{\varphi}_{n-r}(z_i)+P_i\overline{\varphi}_{n+1}(z_l)\qquad(47.12)$$

Consequently, the computation may be carried out on the basis of the same relations as in the case of expression (47.2).

Formula (47.12) is a recurring one, and it may be used to express n-th order moments in any section z_l in terms of moments in an arbitrary section z_0. If, as usual, the origin is taken at section z_0, then, according to (47.2), for section $z_l = z_1$ we obtain

$$\frac{M_n(z_1)}{n!}=\sum_{r=0}^{3}\frac{M_r(0)}{r!}\,\varphi_{n-r}(z_1)+P_1\varphi_{n+1}(z_1)$$

For any section z_l

$$\frac{\overline{M}_n(z_i)}{n!}=\sum_{s=0}^{3}\frac{M_s(0)}{s!}\,\frac{a_s^n(z_i)}{n!}+\sum_{k=1}^{i}P_k\,\frac{b_k^n(z_l)}{n!}\qquad(47.13)$$

where

$$\left.\begin{aligned}\frac{a_s^n(z_i)}{n!}&=\sum_{r=0}^{3}\frac{\beta_i^r}{r!}\cdot\frac{a_s^r(l_{l-1})}{r!}\,\overline{\varphi}_{n-r}(z_l)\\ \frac{b_k^n(z_l)}{n!}&=\sum_{r=0}^{3}\frac{\beta_i^r}{r!}\cdot\frac{b_k^r(l_{l-1})}{r!}\,\overline{\varphi}_{n-r}(z_i)\\ \frac{b_k^n(z_k)}{n!}&=\overline{\varphi}_{n+1}(z_k)\end{aligned}\right\}\qquad(47.14)$$

By using formula (47.13) we shift from one section to another until we reach the right-hand support. Then the expressions for the n-th moments in sections $z_1 = 0$ and $z_l = z_m$ will be used to find the boundary conditions for the given beam. These boundary conditions will be written in the form of (46.19) when $j = 1$.

For convenience of calculation let us change the form of the function $\overline{\varphi}_{n-r}(z_i)$ in formulas (47.14). According to (47.6) and (47.7), when $n = 3$

$$\left.\begin{aligned}
&\bar{\varphi}_0(z_i)=\varphi_0(z_i)-\gamma\varphi_{-2}(z_i)-\Pi_i\left[\varphi_3(z_i)-\gamma_i\varphi_1(z_i)\right]\\
&\bar{\varphi}_1(z_i)=\varphi_1(z_i)-\gamma_i\varphi_{-1}(z_i)\\
&\bar{\varphi}_2(z_i)=\varphi_2(z_i);\quad \bar{\varphi}_3(z_i)=\varphi_3(z_i)-\gamma_i\varphi_1(z_1)\\
&\text{when } n=2\\
&\bar{\varphi}_{-1}(z_i)=\varphi_{-1}(z_i)-\Pi_i\varphi_2(z_i)\\
&\bar{\varphi}_0(z_i)=\bar{\varphi}_0(z_i);\quad \bar{\varphi}_1(z_i)=\varphi_1(z_i);\quad \bar{\varphi}_2(z_i)=\varphi_2(z_i)\\
&\text{when } n=1\\
&\bar{\varphi}_{-2}(z_i)=\varphi_{-2}(z_i)-\Pi_i\varphi_1(z_i)\\
&\bar{\varphi}_{-1}(z_i)=\varphi_{-1}(z_i);\quad \bar{\varphi}_0(z_i)=\varphi_0(z_i);\quad \bar{\varphi}_1(z_i)=\varphi_1(z_i)\\
&\text{when } n=0\\
&\bar{\varphi}_{-3}(z_i)=\varphi_{-3}(z_i)-\Pi_i\varphi_0(z_i)\\
&\bar{\varphi}_{-2}(z_i)=\varphi_{-2}(z_i);\quad \bar{\varphi}_{-1}(z_i)=\varphi_{-1}(z_i)\\
&\bar{\varphi}_0(z_i)=\varphi_0(z_i)
\end{aligned}\right\}\quad(47.15)$$

Functions $\bar{\varphi}_{n+1}(z_i)$ have the form

$$\left.\begin{aligned}
&\bar{\varphi}_4(z_i)=\varphi_4(z_i)-\gamma_i\varphi_2(z_i)\\
&\bar{\varphi}_3(z_i)=\varphi_3(z_i);\quad \bar{\varphi}_2(z_i)=\varphi_2(z_i);\quad \bar{\varphi}_1(z_i)=\varphi_1(z_i)
\end{aligned}\right\}\quad(47.16)$$

where

$$\Pi_l=\frac{1}{A_{l-1}EI_l}\qquad(47.17)$$

Functions $\varphi_i(z)$ may be expressed by Puzyrevsky's functions $V_i(z)$.

If the functions included in (47.15) and (47.16) are determined by equation $\varphi_i^{IV}(z)=-4\rho^4\varphi_i(z)$, then the functions $\varphi_i(z)$ and Puzyrevsky's functions have the following relationship

$$\begin{aligned}
&\varphi_0(z)=V_0(z);\quad \varphi_3(z)=\frac{1}{2\sqrt{2}\rho^3}V_3(z)\\
&\varphi_1(z)=\frac{1}{v_2\rho}V_1(z);\quad \varphi_4(z)=-\frac{1}{4\rho^4}\left[V_0(z)-1\right]\\
&\varphi_2(z)=\frac{1}{2\rho^2}V_2(z)\\
&\varphi_{-1}(z)=-V_2\rho V_3(z);\quad \varphi_{-3}(z)=-2V_2\rho^3V_1(z)\\
&\varphi_{-2}(z)=-2\rho^2V_2(z);\quad \varphi_{-4}(z)=-4\rho^4V_0(z)
\end{aligned}\qquad(47.18)$$

In deriving formulas (47.13) and (47.14), the moment of inertia of the beam, as well as the loads and the rigidity of the elastic foundation, were assumed constant within the span between the elastic supports. If this assumption leads to a considerable error in the solutions, then such a span should be divided into several parts and the previous formulas applied by assuming the intermediate supports to have infinite pliability, i. e., $A_i \to \infty$.

Formula (47.13) provides a convenient tabular form of calculations. The procedure is shown in Table 47.1, where the following designations are introduced for brevity

$$\frac{a_s^n(l_i)}{n!} = a_s^{(n)}(l_i); \quad \frac{b_s^n(l_i)}{n!} = b_s^{(n)}(l_i)$$

These coefficients are determined as a sum of the products of the quantities in the row and in the column intersecting at the required coefficient.

For example, coefficient $a_1^{(2)}(l_2)$ is found by adding the products of the quantities in the row beginning with $\frac{M_2(l_2)}{2!}$ and in the column beginning with $\frac{M_1(0)}{1!}$, the extreme left quantity of the row being multiplied by the first top quantity of the column, etc.

After finding the moments $\frac{M_n(0)}{n!}$ from the boundary conditions, the parameters of flexure in any section may be determined from Table 47.1, in which l_i should be replaced by z_i.

48. STABILITY AND VIBRATION OF IRREGULAR DECKS

Let us examine the problem of deck stability. Regular decks are fully discussed in numerous textbooks and handbooks on shipbuilding.

Let us design an irregular deck subjected to longitudinal forces T_j compressing the j-th longitudinal brace, and find the critical value of this force.

In order to compose the expression for the n-th moment, the stability differential equation for a prismatic beam and its solution should be examined. The buckling shape of such a beam $U(z)$ will be represented by a sum of functions $U_i(\alpha z)$.

$$U(z) = \sum_{i=0}^{3} H_i U_i(\alpha z) \tag{48.1}$$

which are determined from equations

$$U_i^{IV}(\alpha z) = -\alpha^2 U_i''(\alpha z) \tag{48.2}$$

where

$$\alpha^2 = \frac{T}{EI} \tag{48.3}$$

and H_i are arbitrary constants.

When $z=0$ these functions obtain the following values

$$\left.\begin{array}{llll} U_0(0)=1; & U_0'(0)=0; & U_0''(0)=0; & U_0'''(0)=0 \\ U_1(0)=0; & U_1'(0)=1; & U_1''(0)=0; & U_1'''(0)=0 \\ U_2(0)=0; & U_2'(0)=0; & U_2''(0)=1; & U_2'''(0)=0 \\ U_3(0)=0; & U_3'(0)=0; & U_3''(0)=0; & U_3'''(0)=1 \end{array}\right\} \qquad (48.4)$$

And, finally,

$$\left.\begin{array}{l} U_0(\alpha z)=1; \quad U_1(\alpha z)=z \\ U_2(\alpha z)=\dfrac{1-\cos\alpha z}{\alpha^2} \\ U_3(\alpha z)=\dfrac{\alpha z-\sin\alpha z}{\alpha^3} \end{array}\right\} \qquad (48.5)$$

According to the general procedure of designing non-prismatic beams, we divide the beam into a number of spans within which the beam may be considered prismatic, and then, similarly to (47.8), the following expression for the n-th order moment of the j-th longitudinal brace in span $(i-1)$, i, may be composed

$$\frac{M_{nj}(z_{ji})}{n!}=\sum_{r=0}^{3}\frac{\beta_{ji}^{r}}{r!}\cdot\frac{M_{rj}(l_{j,\,i-1})}{r!}\cdot U_{3-r}^{3-n}(\alpha_{ji}z_i)\,\frac{\delta_{r0}^{n}}{n!} \qquad (48.6)$$

where expression $\frac{\delta_{r0}^{n}}{n!}$, according to formula (47.6), is equal to

$$U_{3-r}^{3-n}(\alpha_{ji}z_i)\,\frac{\delta_{r0}^{n}}{n!}=U_{3-r}(\alpha_{ji}z_i)-\gamma_i U_{3-r}''(\alpha_{ji}z_i),$$

when $n=3$, $r\neq 1$

$$U_{3-r}^{3-n}(\alpha_{ji}z_i)\,\frac{\delta_{r0}^{n}}{n!}=U_{3-r}^{3-n}(\alpha_{ji}z_i), \text{ otherwise} \qquad (48.7)$$

The top index $(3-n)$ of the function $U(\alpha z)$ gives the order of the derivative, and the coefficient α_{ji} of the argument of this function is equal to

$$\alpha_{ji}=\sqrt{\frac{T_{ji}}{EI_{ji}}}$$

If a reaction $R_{j,i-1}$ is applied at the beginning of span $i-1$, i of the j-th longitudinal brace, then the n-th moment is expressed according to formula (47.9).

$$\frac{\overline{M}_{nj}(z_{ji})}{n!}=\frac{M_{nj}(z_{ji})}{n!}+R_{j,\,i-1}U_3^{3-n}(\alpha_{ji}z_i)\,\frac{\delta_{00}^{n}}{n!} \qquad (48.8)$$

The interaction R_{ji} of the j-th longitudinal brace is given by relation (46.11) in which Q_i must be assumed zero for the stability design.

				$\frac{\beta_i^n}{n!}\,\bar{\varphi}_{n-r}(l_i)$			
i	l_i	n	$\frac{M_n(l_i)}{n!}$	$r=3$	$r=2$	$r=1$	$r=0$
1	l_1	3	$\frac{M_3(l_1)}{3!}$	—	—	—	—
		2	$\frac{M_2(l_1)}{2!}$	—	—	—	—
		1	$\frac{M_1(l_1)}{1!}$	—	—	—	—
		0	$\frac{M_0(l_1)}{0!}$	—	—	—	—
2	l_2	3	$\frac{M_3(l_2)}{3!}$	$\frac{I_2}{I_1}\bar{\varphi}_0(l_2)$	$\frac{I_2}{I_1}\bar{\varphi}_1(l_2)$	$\bar{\varphi}_2(l_2)$	$\bar{\varphi}_3(l_2)$
		2	$\frac{M_2(l_2)}{2!}$	$\frac{I_2}{I_1}\bar{\varphi}_1(l_2)$	$\frac{I_2}{I_1}\bar{\varphi}_0(l_2)$	$\bar{\varphi}_1(l_2)$	$\bar{\varphi}_2(l_2)$
		1	$\frac{M_1(l_2)}{1!}$	$\frac{I_2}{I_1}\bar{\varphi}_2(l_2)$	$\frac{I_2}{I_1}\bar{\varphi}_{-1}(l_2)$	$\bar{\varphi}_0(l_2)$	$\varphi_1(l_2)$
		0	$\frac{M_0(l_2)}{0!}$	$\frac{I_2}{I_1}\bar{\varphi}_3(l_2)$	$\frac{I_2}{I_1}\bar{\varphi}_{-2}(l_2)$	$\bar{\varphi}_{-1}(l_2)$	$\varphi_0(l_2)$
…	…	…	…	…	…	…	…
$m-1$	l_{m-1}	3	$\frac{M_3(l_{m-1})}{3!}$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_0(l_{m-1})$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_1(l_{m-1})$	$\bar{\varphi}_2(l_{m-1})$	$\bar{\varphi}_3(l_{m-1})$
		2	$\frac{M_2(l_{m-1})}{2!}$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_{-1}(l_{m-1})$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_0(l_{m-1})$	$\bar{\varphi}_1(l_{m-1})$	$\bar{\varphi}_2(l_{m-1})$
		1	$\frac{M_1(l_{m-1})}{1!}$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_{-2}(l_{m-1})$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_{-1}(l_{m-1})$	$\bar{\varphi}_0(l_{m-1})$	$\bar{\varphi}_1(l_{m-1})$
		0	$\frac{M_0(l_{m-1})}{0!}$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_{-3}(l_{m-1})$	$\frac{I_{m-1}}{I_{m-2}}\bar{\varphi}_{-2}(l_{m-1})$	$\bar{\varphi}_{-1}(l_{m-1})$	$\bar{\varphi}_0(l_{m-1})$

Table 47.1

$\frac{M_s(0)}{s!}$				P_l				
$\frac{M_0(0)}{0!}$	$\frac{M_1(0)}{1!}$	$\frac{M_2(0)}{2!}$	$\frac{M_3(0)}{3!}$	P_1	P_2	...	P_{m-1}	P_m
$\bar{\varphi}_3(l_1)$	$\bar{\varphi}_2(l_1)$	$\bar{\varphi}_1(l_1)$	$\bar{\varphi}_0(l_1)$	$\bar{\varphi}_4(l_1)$				
$\bar{\varphi}_2(l_1)$	$\bar{\varphi}_1(l_1)$	$\bar{\varphi}_0(l_1)$	$\bar{\varphi}_{-1}(l_1)$	$\bar{\varphi}_3(l_1)$				
$\bar{\varphi}_1(l_1)$	$\bar{\varphi}_0(l_1)$	$\bar{\varphi}_{-1}(l_1)$	$\bar{\varphi}_{-2}(l_1)$	$\bar{\varphi}_2(l_1)$				
$\bar{\varphi}_0(l_1)$	$\bar{\varphi}_{-1}(l_1)$	$\bar{\varphi}_{-2}(l_1)$	$\bar{\varphi}_{-3}(l_1)$	$\bar{\varphi}_1(l_1)$				
$a_0^{(3)}(l_2)$	$a_1^{(3)}(l_2)$	$a_1^{(3)}(l_2)$	$a_3^{(3)}(l_2)$	$b_1^{(3)}(l_2)$	$\bar{\varphi}_4(l_2)$			
$a_0^{(2)}(l_2)$	$a_1^{(2)}(l_2)$	$a_2^{(2)}(l_2)$	$a_2^{(2)}(l_2)$	$b_1^{(2)}(l_2)$	$\bar{\varphi}_3(l_2)$			
$a_0^{(1)}(l_2)$	$a_1^{(1)}(l_2)$	$a_2^{(1)}(l_2)$	$a_1^{(1)}(l_2)$	$b_1^{(1)}(l_2)$	$\bar{\varphi}_2(l_2)$			
$a_0(l_2)$	$a_1(l_2)$	$a_2(l_2)$	$a_0(l_2)$	$b_1(l_2)$	$\bar{\varphi}_1(l_2)$			
...	...	...	...	...	...	...	...	...
$a_0^{(3)}(l_{m-1})$	$a_1^{(3)}(l_{m-1})$	$a_2^{(3)}(l_{m-1})$	$a_3^{(3)}(l_{m-1})$	$b_1^{(3)}(l_{m-1})$	$b_2^{(3)}(l_{m-1})$		$\bar{\varphi}_4(l_{m-1})$	
$a_0^{(2)}(l_{m-1})$	$a_1^{(2)}(l_{m-1})$	$a_2^{(2)}(l_{m-1})$	$a_3^{(2)}(l_{m-1})$	$b_1^{(2)}(l_{m-1})$	$b_2^{(2)}(l_{m-1})$		$\bar{\varphi}_3(l_{m-1})$	
$a_0^{(1)}(l_{m-1})$	$a_1^{(1)}(l_{m-1})$	$a_2^{(1)}(l_{m-1})$	$a_3^{(1)}(l_{m-1})$	$b_1^{(1)}(l_{m-1})$	$b_2^{(1)}(l_{m-1})$		$\bar{\varphi}_2(l_{m-1})$	
$a_0(l_{m-1})$	$a_1(l_{m-1})$	$a_2(l_{m-1})$	$a_3(l_{m-1})$	$b_1(l_{m-1})$	$b_2(l_{m-1})$		$\bar{\varphi}_1(l_{m-1})$	

				$\frac{\beta_i^n}{n!}\bar{\varphi}_{n-r}(l_i)$			
i	l_i	n	$\frac{M_n(l_m)}{n!}$	$r=3$	$r=2$	$r=1$	$r=0$
m	l_m	3	$\frac{M_3(l_m)}{3!}$	$\frac{I_m}{I_{m-1}}\varphi_0(l_m)$	$\frac{I_m}{I_{m-1}}\bar{\varphi}_1(l_m)$	$\bar{\varphi}_2(l_m)$	$\bar{\varphi}_3(l_m)$
		2	$\frac{M_2(l_m)}{2!}$	$\frac{I_m}{I_{m-1}}\varphi_{-1}(l_m)$	$\frac{I_m}{I_{m-1}}\bar{\varphi}_0(l_m)$	$\bar{\varphi}_1(l_m)$	$\bar{\varphi}_2(l_m)$
		1	$\frac{M_1(l_m)}{1!}$	$\frac{I_m}{I_{m-1}}\varphi_{-2}(l_m)$	$\frac{I_m}{I_{m-1}}\bar{\varphi}_{-1}(l_m)$	$\bar{\varphi}_0(l_m)$	$\bar{\varphi}_1(l_m)$
		0	$\frac{M_0(l_m)}{0!}$	$\frac{I_m}{I_{m-1}}\varphi_{-3}(l_m)$	$\frac{I_m}{I_{m-1}}\bar{\varphi}_{-2}(l_m)$	$\bar{\varphi}_{-1}(l_m)$	$\bar{\varphi}_0(l_m)$

Elimination of the unknown reaction from expression (48.8) results in a recurrent formula for the n-th order moment

$$\frac{\bar{M}_{nj}(z_{ji})}{n!}=\sum_{r=0}^{3}\frac{\bar{M}_{rj}(l_{j,\,i-1})}{r!}\cdot\frac{\beta_{ji}^{r}}{r!}U_{3-r}^{3-n}(\alpha_{ji}z_i)\frac{\delta_{r0}^{n}}{n!}+$$

$$+U_3^{3n}(\alpha_{ji}z_i)\frac{\delta_{r0}^{n}}{n!}\sum_{q=1}^{i}S_{j,\,q,\,i-1}\frac{\bar{M}_{3q}(l_{q,\,i-1})}{3!} \tag{48.9}$$

Let us express the moments in an arbitrary section z_{ji} in terms of moments at the origin.

When $z_{ji}=z_{j1}$,

$$\frac{\bar{M}_{nj}(z_{j1})}{n!}=\sum_{r=0}^{3}\frac{\bar{M}_{rj}(0)}{r!}\cdot\frac{\beta_{ji}^{r}}{r!}U_{3-r}^{3-n}(\alpha_{j1}z_1)\frac{\delta_{r0}^{n}}{n!} \tag{48.10}$$

When $z_{ji}\geqslant z_{j2}$

$$\frac{\bar{M}_{nj}(z_{ji})}{n!}=\sum_{q=1}^{i}\sum_{s=0}^{3}\frac{M_{sq}(0)}{s!}\cdot\frac{D_{qn}^{s}(\alpha_{ji}z_i)}{s!} \tag{48.11}$$

Continued

$\frac{M_s(0)}{s!}$				P_i				
$\frac{M_0(0)}{0!}$	$\frac{M_1(0)}{1!}$	$\frac{M_2(0)}{2!}$	$\frac{M_3(0)}{3!}$	P_1	P_2	…	P_{m-1}	P_m
$a_0^{(3)}(l_m)$	$a_1^{(3)}(l_m)$	$a_2^{(3)}(l_m)$	$a_3^{(3)}(l_m)$	$b_1^{(3)}(l_m)$	$b_{12}^{(3)}(l_m)$		$b_{m-1}^{(3)}(l_m)$	$\bar{\varphi}_4(l_m)$
$a_0^{(2)}(l_m)$	$a_1^{(2)}(l_m)$	$a_2^{(2)}(l_m)$	$a_3^{(2)}(l_m)$	$b_1^{(2)}(l_m)$	$b_2^{(2)}(l_m)$		$b_{m-1}^{(2)}(l_m)$	$\bar{\varphi}_3(l_m)$
$a_0^{(1)}(l_m)$	$a_1^{(1)}(l_m)$	$a_2^{(1)}(l_m)$	$a_3^{(1)}(l_m)$	$b_1^{(1)}(l_m)$	$b_2^{(1)}(l_m)$		$b_{m-1}^{(1)}(l_m)$	$\bar{\varphi}_2(l_m)$
$a_0(l_m)$	$a_1(l_m)$	$a_2(l_m)$	$a_3(l_m)$	$b_1(l_m)$	$b_2(l_m)$		$b_{m-1}(l_m)$	$\bar{\varphi}_1(l_m)$

$$\left.\begin{aligned}
&\text{and for } z_{jl}=z_{j2}\\
&\frac{D_{qn}^{s}(\alpha_{j2}z_2)}{s!}=S_{j,q,1}U_{3-s}(\alpha_{q1}l_1)\frac{\delta_{s0}^{3}}{3!}U_3^{3-n}(\alpha_{j2}z_2)\frac{\delta_{00}^{n}}{n!}+\\
&+\varepsilon_{jq}\sum_{r=0}^{3}\frac{\beta_{j2}^{r}}{r!}U_{3-r}^{3-n}(\alpha_{j2}z_2)\frac{\delta_{r0}^{n}}{n!}U_{3-s}^{3-r}(\alpha_{j1}l_1)\frac{\delta_{00}^{n}}{r!}\\
&\quad\text{When } z_{jl}\geqslant z_{j3}\\
&\frac{D_{qn}^{s}(\alpha_{jl}z_l)}{s!}=\sum_{r=0}^{3}\frac{\beta_{jl}^{r}}{r!}U_{3-r}^{3-n}(\alpha_{jl}z_l)\frac{\delta_{r0}^{n}}{n!}\times\\
&\times\frac{D_{qr}^{s}(\alpha_{j,l-1}l_{l-1})}{s!}+\sum_{p=1}^{l}S_{j,p,l-1}\frac{D_{q3}^{s}(\alpha_{p,l-1}l_{l-1})}{s!}U_3^{3-n}(\alpha_{jl}z_l)\frac{\delta_{00}^{n}}{n!}
\end{aligned}\right\}\tag{48.12}$$

Suppose the compressing forces T_j acting on each of the j longitudinal braces are proportional to one parameter, i.e.,

$$T_j = T_0\tau_j(z)$$

where T_0 = unknown parameter
$\tau_j(z)$ = function of variation of the compressive forces along the brace

The equation of critical forces, from which the parameter T_0 may be determined, is composed from the boundary conditions (46.19). This equation is easily solved by the remainder method, which makes it possible to use a digital computer.

The buckling shape may be determined from formulas (48.10) and (48.11) by assuming $n = 3$.

Let us now calculate the vibration of the decks. The natural vibrational modes $U_n^{IV}(\bar{\mu}_n z)$ of a prismatic beam are given by the differential equation

$$U_n^{IV}(\bar{\mu}_n z) = \bar{\mu}_n^4 U_n(\mu_n z) \tag{48.13}$$

. . . .

where

$$\bar{\mu}_n = \left(\frac{\mu_n}{l}\right)^4 = \frac{m\lambda_n^2}{EI}$$

m = distributed mass of the beam
λ_n = natural vibration frequency of the n-th mode

Let us express the natural vibrational modes $U_n(\bar{\mu}_n z)$ in terms of functions $U_i(\bar{\mu}_n z)$ which, together with their corresponding derivatives, satisfy the unit matrix of form (48.4) at the origin of coordinates

$$U_n(\bar{\mu}_n z) = \sum_{i=0}^{3} U_i(\mu_n z) H_i^*$$

These functions are determined by introducing a normalizing factor and have the following form

$$\left.\begin{aligned}
U_0(\bar{\mu}_n z) &= \frac{1}{2}(\cosh \bar{\mu}_n z + \cos \bar{\mu}_n z)\\
U_1(\bar{\mu}_n z) &= \frac{1}{2\bar{\mu}_n}(\sinh \bar{\mu}_n z + \sin \bar{\mu}_n z)\\
U_2(\bar{\mu}_n z) &= \frac{1}{2\bar{\mu}_n^2}(\cosh \bar{\mu}_n z - \cos \bar{\mu}_n z)\\
U_3(\bar{\mu}_n z) &= \frac{1}{2\bar{\mu}_n^3}(\sinh \bar{\mu}_n z - \sin \bar{\mu}_n z)
\end{aligned}\right\} \tag{48.14}$$

Such a choice of functions makes it possible to perform the evaluation of natural deck vibrations on the basis of the same relationships (48.11) and (48.12), as for the stability, with appropriate substitution of functions (48.5) for (48.14). The influence factors $S_{j,q,i}$ in this case are given by the differential equation of the form (48.13) for transverse beams.

Coefficients $\frac{D^{\varepsilon}_{qn}(\alpha z)}{sl}$ in formula (48.11) are determined from a table similar to Table 47.1, therefore, the programming of calculations for digital computers is possible.

Let us now compare the labour-consumption of the moment-operational method with that of other methods based on equating the positive and negative deflections of beams at the points of their intersection (Navier's method).

Suppose the deck has t longitudinal braces and μ transverse beams. Then the moment-operational method will involve $4t$ simultaneous equations, while Navier's method requires $t(2+\mu)$ equations. Therefore, complex constructions can be solved by the moment-operational method by means of digital computers with a low memory capacity.

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